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t-dependent Backgrounds in String Theory

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Approval

The internship report titled "**t-dependent Backgrounds in String Theory**" submitted by **Kazi Fahim Reza Arko**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfilment of the requirements of the internship program.

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1 Introduction

This report is an extension of the paper *Kasner Branes with Arbitrary Signature* by W.A Sabra [1]. Originally the Kasner solutions were a vacuum Einstein solution describing an anisotropic universe covering the **Bianchi Type Universes**. The anisotropy is described using different expansion and contraction rates of the Kasner metric in different spatial directions. In this study we are more interested in non-vacuum solutions, and are Kasner-like but not Kasner solutions. Mostly because of their useful role in toy models in string theory and supergravity. Having the Kasner exponents to be time dependent one might find it to be a viable candidate for Kination where the requirements are to have a rolling potential and $R < 0$. By incorporating the scaler field one deviates from the vacuum setting and enters non-vacuum regime where $T_{\mu\nu} \neq 0$. The connection between the Kasner solutions and the Kination epoch can be found in how both of these describe the cosmological evolution. So, if one generalizes the Kasner solution allowing the exponents to evolve with time effectively may capture the essence of the Kination epoch, where the expansion of the universe is due to the kinetic energy of a rolling scalar field.

In the first part the Kasner metric is introduced and gotten familiarized with. The mathematical tools necessary (differential forms, integral of forms etc) for introducing the flux were covered on a preliminary basis and the rest is built up from there. Using these necessary requirements the results found on the *Kasner Branes with Arbitrary Signature* were reproduced , especially the results of *d-dimensional gravity theory with m-form, 4D Maxwell Field, d-dimensional gravity theory with a dynamical scalar field* and *Brane solutions.*

A metric ansatz was introduced along with the action (5.3), which represents the action of the bosonic fields of many supergravity theories, to go beyond the typical Kasner type universe, different stringy ingredients like dilations, form fields have been introduced. The ansatz 2-form for the flux and the scalar that introduces the non-vacuum state which deviates from the vacuum Kasner. To capture the time-dependent expansion in the Kasner metric, we allowed the Kasner exponents ($\gamma_i(t)$) to evolve with time rather than being constants. We introduced a scalar field $\phi(x, t)$ and observe whether it is a rolling scalar field that would allow the time-dependent exponents to introduce different expansion or contraction at each spatial direction at any given moment. The solutions from this expected to be a new solutions that would take one beyond the typical Bianchi type universes and would essentially be taken as candidates from Supergravity and String theory along with Kination.

2 The Kasner Metric

2.1 Introduction to the Kasner Metric

$$ds^2 = -dt^2 + t^{2p_1}dx_1^2 + t^{2p_2}dx_2^2 + t^{2p_3}dx_3^2 \tag{2.1}$$

$$\Rightarrow ds^2 = dt^2 + \sum_{i=1}^{3} t^{2p_i}dx_1^2 \tag{2.2}$$

The Kasner metric is a solution to Einstein's field equations in vacuum ($T_{\mu\nu} = 0$) that described an anisotropic but homogeneous universe. It is a particular case of a **Bianchi Type I** cosmology and is often used in discussions of the early universe and near singularities (e.g. in the BKL scenario).

In the first tiny fraction of a second after the Big Bang, the universe was likely not perfectly smooth or isotropic. Quantum fluctuations, intense gravity, and lack of thermal equilibrium could have made the universe wildly anisotropic. Kasner-like solutions describe how an early, chaotic, empty universe might have been before matter and radiation filled it. So, Kasner helps model the pre-inflation era or behavior close to cosmological singularities.

Near singularities, the Einstein equation behave differently. Spatial derivatives can become negligible compared to time derivatives. This leads to solutions like Kasner, where each point evolves almost independently, like in a Bianchi I universe.

This is used in the **BKL (Belinski- Khalatnikov- Lifshitz) scenario**[1] , which suggest that near any singularity, spacetime behaves locally chaotic sequence of **Kasner epochs**. Studying anisotropic but homogeneous models like Kasner, helps us understand what solutions are possible, explore how much structure is permitted by GR, observe how matter content affects anisotropy (Kasner is vacuum, but other models can include matter).

To summarize:

We study Kasner not because the universe looks like that today, but because it might have behaved like that in the past.

It suggest that **As we approach a singularity, each point in space behaves like a separate, chaotic, anisotropic universe, evolving through a series of Kasner- like states.**

Near any singularity (like, $t \to 0$), time derivatives dominate over spatial derivatives in Einstein's equations. That means that the universe **decouples at each spatial point**, and the local behavior is governed by **ordinary differential equations**, not partial ones.

[1] The BKL scenario is a theoretical model describing how spacetime behaves near a spacelike singularity, like the Big Bang or the center of a black hole.

Each point evolves like a Bianchi type model[2], especially **Bianchi type IX**, which allows chaotic transitions between Kasner epochs.

2.1.1 Kasner Epochs and Bounces

In the BKL picture, the Universe near a singularity enters a **Kasner epoch**, an anisotropic state where expansions occurs in some directions and contraction in others. Then, due to curvature effects, the **Kasner exponents** "bounce" to a new set of values. This process repeats over and over, unpredictably which leads to **chaotic evolution**.

The chaotic evolution is sometimes called **Mixmaster dynamics**, especially in the **Bianchi IX model**.

It suggests that the generic behavior of spacetime near singularities is not smooth or isotropic, but chaotic and directionally violent. It provides a contrast to the smoothness of FLRW cosmology and gives us a model for the **pre- inflationary universe**. It is one of the most studied classical frameworks for understanding the structure of spacetime singularities.

One can imagine a spinning box where each side can stretch or shrink but every few seconds the stretching direction suddenly changes. That's like the **Kasner "bounce"** at each point in the BKL scenario.

2.2 The Power Law

A power law is any expression of the form:

$$f(t) = A \cdot t^n \tag{2.3}$$

where:

- A is a constant (can be 1).
- n is the **power** (can be positive, negative, fraction , etc.).
- t is the variable (in this case,cosmic time).

so, $f(t) = t^2$ is a power law, also is $f(t) = t^{-1/3}$. But, $f(t) = \ln t$ or $f(t) =^t$ are not power laws, they are logarithmic or exponentials.

[2]In quantum gravity and loop quantum cosmology, the simplest toy models for quantization of spacetime are often based on **Bianchi models**, especially Bianchi I (Kesner-like). Because they are spatially homogeneous, they reduce Einstein's field equations to ordinary differential equations which are tractable to quantize.

Type	Mathematical Form	Example	Used in
Power- law	$a(t) = t^p$	$t^{2/3}$	Kasner metric, matter/ radiation dominated universe
Exponential	$a(t) = e^{Ht}$	e^{3t}	Inflation, de Sitter space

Table 1. Basic Forms of Power-law and Exponential

Comparing how both grow over time for $t > 0$:

Time t	$t^{2/3}$**(power law)**	e^t **(exponential)**
1	1.00	2.718
2	~ 1.59	~ 7.39
5	~ 2.92	~ 148.41
10	~ 4.64	~ 22026

Table 2. Growth Behavior of Power-law and Exponential

Physical Meaning

Power- law Expansion (Kasner, Matter-Dominated etc.)

$$a(t) = t^p \tag{2.4}$$

- Expansion rate slows down over time.
- The Hubble parameter $H = \frac{\dot{a}}{a} = \frac{p}{t}$ decreases with time.
- Typical of:
  - Kasner Metric (anisotropic)
  - Radiation dominated universe: $a(t) \sim t^{1/2}$
  - Matter dominated universe: $a(t) \sim t^{2/3]}$

Feature	Power- law	Exponential
Acceleration	No (decelerates)	Yes
Hubble Horizon	Increases	Fixed or shrinking
Causal Contact	Expands	Can be lost (inflation stretched space too fast)

Table 3. Geometry and Horizons for Power-law and Exponential

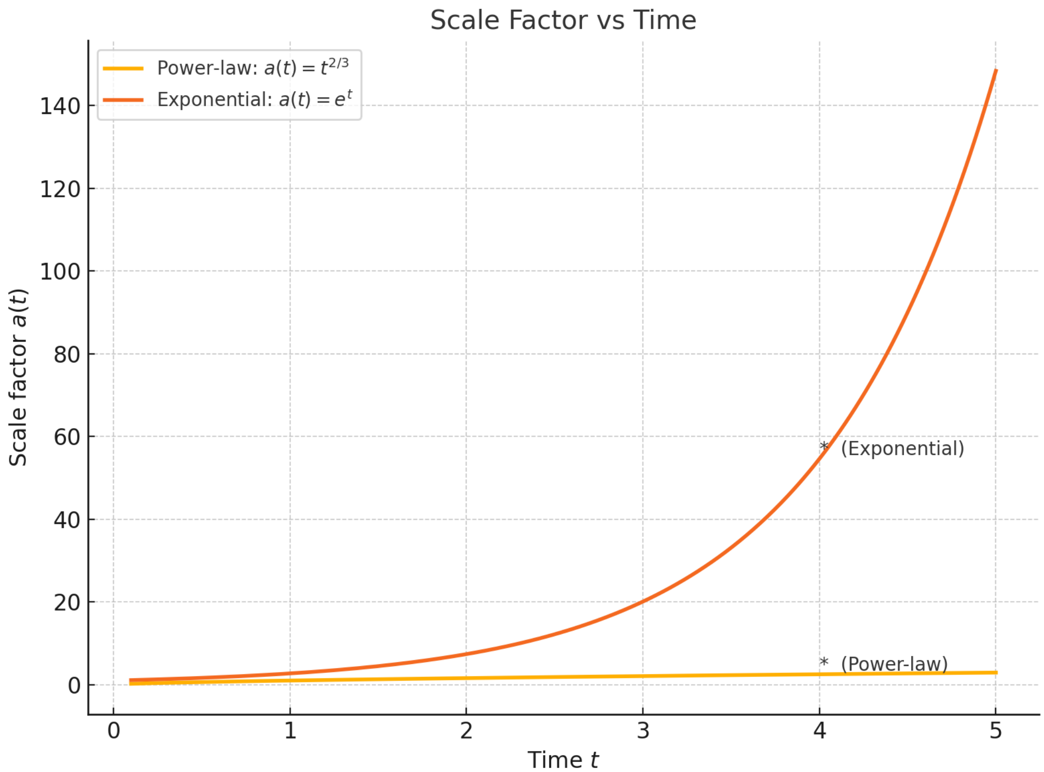

Figure 1. **Scale Factor Vs. Time**

Exponential Expansion (Inflation, de Sitter)

$$a(t) = e^{Ht} \tag{2.5}$$

- Expansion rate is constant: $H = const.$
- Accelerating expansion, distance between points increase faster and faster.
- Typical of
  - Inflationary epoch (early universe).
  - Dark energy- dominated universe (present/ future)
  - de Sitter space

Why use power laws in Kasner metric?

Power laws make the Einstein equations easier to solve. The behavior of space becomes analytically clear, one can immediately see which direction the spacetime is expanding or contracting. Near singularities (like $t \to 0$), power laws describe the **dominant behavior** of the universe.

2.3 Deriving the Kasner Metric

Metric Ansatz:

We begin by assuming the metric of a spatially flat, homogeneous, anisotropic universe:

$$ds^2 = -dt^2 + a_1(t)^2 dx_1^2 + a_2(t)^2 dx_2^2 + a_3(t)^2 dx_3^2 \tag{2.6}$$

This is a special case of a Bianchi Type I metric with three independent scale factors $a_i(t)$. It allows different expansion in each spatial direction.

Assume Power-Law Scale Factors (Kasner Ansatz):

Assuming the scale factors follow **power laws** in time:

$$a_1(t) = t^{p_1},\ a_2(t) = t^{p_2},\ a_3(t) = t^{p_3} \tag{2.7}$$

Then the metric becomes,

$$ds^2 = -dt^2 + t^{2p_1} dx_1^2 + t^{2p_2} dx_2^2 + t^{2p_3} dx_3^2 \tag{2.8}$$

This is our **Kasner Metric**

2.4 The Kasner Metric

$$ds^2 = -dt^2 + t^{2p_1} dx_1^2 + t^{2p_2} dx_2^2 + t^{2p_3} dx_3^2 \tag{2.9}$$

where:

- $t > 0$ is **cosmic time**.
- x, y, z are the **spatial coordinates**.
- p_1, p_2, p_3 are the **Kasner exponents**; real constants that govern how spatial direction expands or contracts with time.

2.4.1 Kasner Conditions (Constraints on the Exponents)

To satisfy Einstein vacuum equations $R_{\mu\nu} = 0$, the exponents must obey:

$$p_1 + p_2 + p_3 = 1 \quad \text{(linear constraint)} \tag{2.10}$$
$$p_1^2 + p_2^2 + p_3^2 = 1 \quad \text{(quadratic constraint)} \tag{2.11}$$

Example:

A famous solution:

$$p_1 = \frac{2}{3},\ p_2 = \frac{2}{3},\ p_3 = -\frac{1}{3} \tag{2.12}$$

The the metric becomes:

$$ds^2 = -dt^2 + t^{4/3}dx_1^2 + t^{4/3}dx_2^2 + t^{-2/3}dx_3^2 \tag{2.13}$$

What happens at $t \to 0$:

This is the **singularity limit**, very early in the universe:

- $t^{2/3} \to 0$: x and y directions **shrink** to zero size.
- $t^{-1/3} \to \infty$: z direction **blows up**, it becomes infinitely stretched.

So, the universe collapses to a line along the z-axis. It's a highly directional collapse (the distance between points gets smaller and smaller until it goes to zero and "collapses"). Only the z-direction survives as $t \to 0$. This is a spacetime singularity, but not isotropic like in FLRW.

What happens as $t \to \infty$:

This is the **far- future** behavior:

- $t^{2/3} \to \infty$: x and y directions **expand forever**.
- $t^{-1/3} \to 0$: z direction **shrinks** to zero.

So in the far future, the universse v=becomes effectively 2-dimensional, flat in the $x - y$ plane and squashed in z.

2.4.2 Computing the Christoffel Symbols

We are working with:

$$ds^2 = -dt^2 + \sum_{i=1}^{3} t^{2p_i} dx_1^2$$

and,

$$g_{\mu\nu} = diag(-1,\ t^{2p_1},\ t^{2p_2},\ t^{2p_3}) \tag{2.14}$$

We are denoting the coordinates as:

$$x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z \tag{2.15}$$

We know, the Christoffel symbol:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{2.16}$$

Now, since $g_{\mu\nu}$ only depends on time t, the only non-zero derivatives are with respect to t. So we only compute:

$$\partial_t g_{11} = \partial_t(t^{2p_1}) = 2p_1 t^{2p_1-1} \tag{2.17}$$

$$\partial_t g_{22} = \partial_t(t^{2p_2}) = 2p_2 t^{2p_2-1} \tag{2.18}$$

$$\partial_t g_{33} = \partial_t(t^{2p_3}) = 2p_3 t^{2p_3-1} \tag{2.19}$$

For the non-zero Christoffels:

1. $\Gamma^0_{11} = \frac{1}{2} g^{0\sigma}(\partial_1 g_{1\sigma} + \partial_1 g_{1\sigma} - \partial_\sigma g_{11}) = \frac{1}{2} g^{00}(-\partial_0 g_{11}) = \frac{1}{2}(-1)(2p_1 t^{2p_1-1}) = p_1 t^{2p_1-1}$
2. $\Gamma^0_{22} = \frac{1}{2} g^{0\sigma}(\partial_2 g_{2\sigma} + \partial_2 g_{2\sigma} - \partial_\sigma g_{22}) = \frac{1}{2} g^{00}(-\partial_0 g_{22}) = \frac{1}{2}(-1)(2p_2 t^{2p_2-1}) = p_2 t^{2p_2-1}$
3. $\Gamma^0_{33} = \frac{1}{2} g^{0\sigma}(\partial_3 g_{3\sigma} + \partial_3 g_{3\sigma} - \partial_\sigma g_{33}) = \frac{1}{2} g^{00}(-\partial_0 g_{33}) = \frac{1}{2}(-1)(2p_3 t^{2p_3-1}) = p_3 t^{2p_3-1}$
4. $\Gamma^1_{01} = \frac{1}{2} g^{1\sigma}(\partial_0 g_{1\sigma} + \partial_1 g_{0\sigma} - \partial_\sigma g_{01}) = \frac{1}{2} g^{11}(\partial_0 g_{11}) = \frac{1}{2}(t^{-2p_1})(2p_1 t^{2p_1-1}) = p_1 t^{-1}$
5. $\Gamma^2_{02} = \frac{1}{2} g^{2\sigma}(\partial_0 g_{2\sigma} + \partial_2 g_{0\sigma} - \partial_\sigma g_{02}) = \frac{1}{2} g^{22}(\partial_0 g_{22}) = \frac{1}{2}(t^{-2p_2})(2p_2 t^{2p_2-1}) = p_2 t^{-1}$
6. $\Gamma^3_{03} = \frac{1}{2} g^{3\sigma}(\partial_0 g_{3\sigma} + \partial_3 g_{0\sigma} - \partial_\sigma g_{03}) = \frac{1}{2} g^{33}(\partial_0 g_{33}) = \frac{1}{2}(t^{-2p_3})(2p_3 t^{2p_3-1}) = p_3 t^{-1}$

So the **non-zero Christoffel symbols** are:

$$\Gamma^0_{11} = p_1 t^{2p_1-1}, \quad \Gamma^0_{22} = p_2 t^{2p_2-1}, \quad \Gamma^0_{33} = p_3 t^{2p_3-1},$$
$$\Gamma^1_{01} = p_1 t^{-1}, \quad \Gamma^2_{02} = p_2 t^{-1}, \quad \Gamma^3_{03} = p_3 t^{-1}$$

2.4.3 Computing the Ricci Tensor ($R\mu\nu$)

$$\boxed{R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\sigma_{\lambda\sigma}\Gamma^\lambda_{\mu\sigma} - \Gamma^\sigma_{\lambda\nu}\Gamma^\lambda_{\mu\sigma}} \tag{2.20}$$

Calculating:

$$R_{00} = \partial_\sigma \Gamma^\sigma_{00} - \partial_0 \Gamma^\sigma_{0\sigma} + \Gamma^\sigma_{\lambda\sigma}\Gamma^\lambda_{0\sigma} - \Gamma^\sigma_{\lambda 0}\Gamma^\lambda_{0\sigma} = -\partial_0 \Gamma^\sigma_{0\sigma} - \Gamma^\sigma_{\lambda 0}\Gamma^\lambda_{0\sigma}$$

Now,

$$\partial_0 \Gamma^1_{01} = \partial_0(p_1 t^{-1}) = -\frac{p_1}{t^2}$$

so, we can write,

$$\partial_0 \Gamma^\sigma_{0\sigma} = -\frac{p_1 + p_2 + p_3}{t^2} \tag{2.21}$$

Now for,

$$\Gamma^{\sigma}_{\lambda\sigma}\Gamma^{\lambda}_{0\sigma}$$

we can write,

$$\Gamma^{i}_{0i}\Gamma^{i}_{0i}$$

for $i = 1, 2, 3$.

$$(\Gamma^{i}_{0i})^2 = \frac{p_1^2 + p_2^2 + p_3^2}{t^2} \tag{2.22}$$

Combining them together,

$$-\partial_0\Gamma^{\sigma}_{0\sigma} - \Gamma^{\sigma}_{\lambda\sigma}\Gamma^{\lambda}_{0\sigma} = \frac{p_1 + p_2 + p_3}{t^2} - \frac{p_1^2 + p_2^2 + p_3^2}{t^2} = \frac{1}{t^2}(p_1 + p_2 + p_3 - p_1^2 + p_2^2 + p_3^2) \tag{2.23}$$

Now, for R_{11}:

$$\begin{aligned} R_{11} &= \partial_\sigma\Gamma^{\sigma}_{11} - \partial_1\Gamma^{\sigma}_{1\sigma} + \Gamma^{\sigma}_{\lambda\sigma}\Gamma^{\lambda}_{11} - \Gamma^{\sigma}_{\lambda 1}\Gamma^{\lambda}_{1\sigma} \\ \Rightarrow R_{11} &= \partial_\sigma\Gamma^{\sigma}_{11} + \Gamma^{\sigma}_{\lambda\sigma}\Gamma^{\lambda}_{11} - \Gamma^{\sigma}_{\lambda 1}\Gamma^{\lambda}_{1\sigma} \end{aligned} \tag{2.24}$$

Now,

$$\begin{aligned} &\partial_\sigma\Gamma^{\sigma}_{11} = \partial_0\Gamma^{0}_{11} = \partial_0(p_1 t^{2p_1-1}) = p_1(2p_1-1)t^{2(p_1-1)} \\ &\Gamma^{\sigma}_{\lambda\sigma}\Gamma^{\lambda}_{11} = \Gamma^{0}_{11}\Gamma^{i}_{0i} = p_1 t^{2p_1-1}t^{-1}(p_1+p_2+p_3) = p_1(p_1+p_2+p_3)t^{2(p_1-1)} \\ &\Gamma^{\sigma}_{\lambda 1}\Gamma^{\lambda}_{1\sigma} = \Gamma^{0}_{11}\Gamma^{1}_{01} = 2p_1 t^{2p_1-1}p_1 t^{-1} = 2p^2 t^{2(p_1-1)} \end{aligned} \tag{2.25}$$

so,

$$\begin{aligned} R_{11} &= p_1(2p_1-1)t^{2(p_1-1)} + p_1(p_1+p_2+p_3)t^{2(p_1-1)} - 2p_1^2 t^{2(p_1-1)} \\ &= t^{2(p_1-1)}(p_1(2p_1-1) + p_1(p_1+p_2+p_3) - 2p_1^2) \\ &= p_1(p_1+p_2+p_3-1)t^{2(p_1-1)} \end{aligned} \tag{2.26}$$

Similarly we get,

$$R_{22} = p_2(p_1+p_2+p_3-1)t^{2(p_2-1)} \tag{2.27}$$

$$R_{33} = p_3(p_1+p_2+p_3-1)t^{2(p_3-1)} \tag{2.28}$$

We then have:

$$R_{00} = \frac{1}{t^2}(p_1 + p_2 + p_3 - p_1^2 + p_2^2 + p_3^2) \tag{2.29}$$
$$R_{11} = p_1(p_1 + p_2 + p_3 - 1)t^{2(p_1-1)} \tag{2.30}$$
$$R_{22} = p_2(p_1 + p_2 + p_3 - 1)t^{2(p_2-1)} \tag{2.31}$$
$$R_{33} = p_3(p_1 + p_2 + p_3 - 1)t^{2(p_3-1)} \tag{2.32}$$

Now, since they are meant to be the solutions to Einstein's vacuum equation, $R_{\mu\nu} = 0$, we can see that for each of the Ricci tensors to be zero in (2.29). The terms inside, $(p_1 + p_2 + p_3)$ has to equal to $(p_1^2 + p_2^2 + p_3^2)$.

From the R_{ii} (2.30) (2.31) (2.32) Ricci tensors we get,

$$p_1 + p_2 + p_3 = 1 \tag{2.33}$$

(In order to satissfy $R_{ii} = 0$) and, since $(p_1 + p_2 + p_3) = (p_1^2 + p_2^2 + p_3^2)$, we get:

$$(p_1 + p_2 + p_3) = 1 \tag{2.34}$$
$$(p_1^2 + p_2^2 + p_3^2) = 1. \tag{2.35}$$

These are the **Kasner Conditions** mentioned above (2.10) (2.11). Using these conditions we can get the values for the **Kasner Exponents**, such as:

$$p_1 = \frac{2}{3},\ p_2 = \frac{2}{3},\ p_3 = -\frac{1}{3} \tag{2.36}$$

Then:

- $a_x(t) = t^{2/3} \longrightarrow$ **expansion**
- $a_y(t) = t^{2/3} \longrightarrow$ **expansion**
- $a_z(t) = t^{-1/3} \longrightarrow$ **contraction**

so, the x and y directions expand as time increases and the z direction shrinks, which is a **direction-dependent** behavior $\rightarrow$ **Anisotropy**.

Therefore, we have:

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \quad \text{with} \begin{cases} p_1 + p_2 + p_3) = 1 \\ \\ (p_1^2 + p_2^2 + p_3^2) = 1 \end{cases} \tag{2.37}$$

Now for $p_1 = p_2 = p_3 = 0$, the Kasner metric boils down to:

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \tag{2.38}$$

which is the Minkowski metric for flat spcaetime.

Moreover, for $p_i \to \infty$, there are no finite solutions. The valid Kasner exponents lie on the intersection of a **plane** and a **sphere**:

- The Plane:

$$p_1 + p_2 + p_3 = 1 \tag{2.39}$$

- The Sphere:

$$p_1^2 + p_2^2 + p_3^2 = 1 \tag{2.40}$$

The intersection forms a circle, the **Kasner Circle**. So, there are no points at infinity in the space of allowed Kasner exponents.

2.5 The Kasner Metric in 10D

Using the mathematica code:

```
ClearAll["Global`*"]; n = 10;
coord = Join[{t}, Table[Symbol["x" <> ToString[i]], {i, 1, 9}]];
p = Table[Symbol["p" <> ToString[i]], {i, 1, 9}];
metric = DiagonalMatrix[Join[{-1}, Table[t^(2 p[[i]]), {i, 1, 9}]]];
inversemetric = Simplify[Inverse[metric]];

christ[a_, b_, c_] := christ[a, b, c] = Simplify[
  Sum[(1/2) * inversemetric[[a, d]] *
    (D[metric[[d]][[c]], coord[[b]]] +
     D[metric[[d]][[b]], coord[[c]]] -
     D[metric[[b]][[c]], coord[[d]]]), {d, 1, n}]
];

Print["--- Non-zero Christoffel Symbols ---"];
Table[Module[{val = christ[,  ,   ]},
  If[val =!= 0, Print[Subsuperscript["", Row[{,  }],   ], " = ", val]]],
  {  , 1, n}, { , 1, n}, { , 1, n}
];

ricci[ _ ,  _ ] := ricci[ ,   ] = Simplify[
  Sum[D[christ[ ,   ,   ], coord[[ ]]] -
     D[christ[ ,   ,   ], coord[[ ]]] +
     Sum[christ[ ,   ,   ]*christ[ ,   ,   ] -
        christ[ ,   ,   ]*christ[ ,   ,   ], {  , 1, n}], { , 1, n}]
];

Print["--- Non-zero Ricci Tensor Components ---"];
Table[Module[{val = ricci[,  ]},
  If[val =!= 0, Print[Subscript["R", Row[{, }]], " = ", val]]],
  {  , 1, n}, { , 1, n}
];
```

The non zero Christoffel Symbols are:

$$\Gamma^0_{11} = p_1 t^{2p_1-1}, \quad \Gamma^0_{22} = p_2 t^{2p_2-1}, \Gamma^0_{33} = p_3 t^{2p_3-1}, \quad \Gamma^0_{44} = p_4 t^{2p_4-1}, \Gamma^0_{55} = p_5 t^{2p_5-1}, \quad \Gamma^0_{66} = p_6 t^{2p_6-1},$$
$$\Gamma^0_{77} = p_7 t^{2p_7-1}, \quad \Gamma^0_{88} = p_8 t^{2p_8-1}, \Gamma^0_{99} = p_9 t^{2p_9-1}, \quad \Gamma^1_{01} = \Gamma^1_{10} = \frac{p_1}{t}, \Gamma^2_{02} = \Gamma^2_{20} = \frac{p_2}{t}, \quad \Gamma^3_{03} = \Gamma^3_{30} = \frac{p_3}{t},$$
$$\Gamma^4_{04} = \Gamma^4_{40} = \frac{p_4}{t}, \quad \Gamma^5_{05} = \Gamma^5_{50} = \frac{p_5}{t}, \Gamma^6_{06} = \Gamma^6_{60} = \frac{p_6}{t}, \quad \Gamma^7_{07} = \Gamma^7_{70} = \frac{p_7}{t}, \Gamma^8_{08} = \Gamma^1_{80} = \frac{p_8}{t}, \quad \Gamma^9_{09} = \Gamma^9_{90} = \frac{p_9}{t}$$

The non zero Ricci Tensors are:

$$R_{00} = \frac{[(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9) - (p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2 + p_7^2 + p_8^2 + p_9^2)]}{t^2} \tag{2.41}$$

$$R_{11} = p_1(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 - 1)t^{2p_1-2}$$
$$R_{22} = p_2(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 - 1)t^{2p_2-2}$$
$$R_{33} = p_3(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 - 1)t^{2p_3-2}$$
$$R_{44} = p_4(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 - 1)t^{2p_4-2}$$
$$R_{55} = p_5(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 - 1)t^{2p_5-2}$$
$$R_{66} = p_1(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 - 1)t^{2p_6-2}$$
$$R_{77} = p_7(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 - 1)t^{2p_7-2}$$
$$R_{88} = p_8(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 - 1)t^{2p_8-2}$$
$$R_{99} = p_9(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 - 1)t^{2p_9-2}$$

So in general,

$$R_{00} = \frac{1}{t^2}\left(\sum_{i=1}^{9} p_i + \sum_{i=1}^{9} p_i^2\right) \tag{2.42}$$
$$R_{ij} = p_i\left(\sum_{j=1}^{9} p_j - 1\right) t^{2p_i-2} \tag{2.43}$$

3 Varying the actions from Sabra's paper

Equation 1.10

$$S = \int d^d x \sqrt{|g|}\ (R - \frac{\epsilon}{2m!} F_m^2)$$

In forms notation:

$$\int d^d x \sqrt{|g|}\left(R - \frac{\epsilon}{2m!} F_m^2\right) = \int \sqrt{|g|}\left(R \star 1 - \frac{\epsilon}{2} F_m \wedge \star F_m\right) \tag{3.1}$$

Where in (3.1), $F_m = dA_{m-1}$. A_{m-1} is a potential of $(m-1)$- form and dA_{m-1} gives us m-form.

Now, we vary with respect to A_{m-1} and use the principle of least action:

$$\delta_{A_{m-1}} S = 0 \tag{3.2}$$

Applying (3.2) in our (3.1) :

$$\delta S = \int \delta(R \star 1) - \int \delta\left(\frac{\epsilon}{2} F_m \wedge \star F_m\right) = 0 \tag{3.3}$$

Now, for the first term:

$$\delta(R \star 1) = 0 \tag{3.4}$$

since, dependent on the metric.
So we should have:

$$\begin{aligned} \delta_{A_{m-1}} S &= \int \delta\left(-\frac{\epsilon}{2} F_m \wedge \star F_m\right) = 0 \\ \Rightarrow -\frac{\epsilon}{2} &\int (\delta F_m \wedge \star F_m + F_m \wedge \star \delta F_m) = 0 \end{aligned} \tag{3.5}$$

and,

$$\delta F_m \wedge \star F_m = F_m \wedge \star \delta F_m \tag{3.6}$$

So, we have:

$$\begin{aligned} \delta S &= -\epsilon \int (\delta F_m \wedge \star F_m) \\ &= -\epsilon \int \delta(dA_{m-1}) \wedge \star F_m \end{aligned} \tag{3.7}$$

From the graded Leibniz rule:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

$$\Rightarrow d\alpha \wedge \beta = d(\alpha \wedge \beta) - (-1)^p \alpha \wedge \beta$$

So we have,

$$
\begin{aligned}
dA_{m-1} \wedge \star F_m &= d\delta A_{m-1} \wedge \star F_m \\
&= d(\delta A_{m-1} \wedge \star F_m) - (-1)^{m-1} \delta A_{m-1} \wedge d \star F_m
\end{aligned} \tag{3.8}
$$

Now,

$$
\begin{aligned}
&\int_M d(\delta A_{m-1}) \wedge \star F_m \\
&= \int_{\partial M} \delta A_{m-1} \wedge \star F_m - (-1)^{m-1} \int_M \delta A_{m-1} \wedge d \star F_m
\end{aligned} \tag{3.9}
$$

With appropriate boundary conditions on (3.9), we have:

$$
\begin{aligned}
\delta_{A_{m-1}} S &= \epsilon (-1)^{m-1} \int_M \delta A_{m-1} \wedge d \star F_m \\
&= \epsilon \int_M \delta A_{m-1} \wedge (-1)^{m-1} d \star F_m
\end{aligned} \tag{3.10}
$$

and,

$$
d \star F_m = 0 \tag{3.11}
$$

$$
\Rightarrow \nabla_\mu (F^{\mu\nu_2 \ldots \nu_m}) = 0 \tag{3.12}
$$

$$
\Rightarrow \partial_\mu (\sqrt{|g|} F^{\mu\nu_2 \ldots \nu_m}) = 0 \tag{*}
$$

Now, we vary w.r.t the metric:
From (3.1) we have:

$$
\int d^d x \biggl(-\frac{\epsilon}{2m!} F_m^2 \sqrt{|g|} \biggr) \tag{3.13}
$$

where, $F_m^2 = F_{\mu_1 \ldots \mu_m} F^{\mu_1 \ldots \mu_m}$.
Now,

$$
\begin{aligned}
&\int \delta \biggl(-\frac{\epsilon}{2m!} F_m^2 \sqrt{|g|} \biggr) \\
&= -\frac{\epsilon}{2m!} \int \biggl(\delta F_m^2 \sqrt{|g|} + F_m^2 \delta \sqrt{|g|} \biggr)
\end{aligned} \tag{3.14}
$$

We know that,

$$
\delta \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu} \tag{3.15}
$$

and,

$$\begin{aligned}\delta F_m^2 &= \delta(F_{\mu_1\ldots\mu_m}F^{\mu_1\ldots\mu_m})\\ &= \delta(g^{\mu_1\nu_1}\ldots g^{\mu_m\nu_m}F_{\mu_1\ldots\mu_m}F_{\nu_1\ldots\nu_m})\\ &= \delta g^{\mu_1\nu_1}g^{\mu_2\nu_2}\ldots g^{\mu_m\nu_m}F_{\mu_1\ldots\mu_m}F_{\nu_1\ldots\nu_m}\end{aligned} \tag{3.16}$$

Considering $m=2$:

$$\begin{aligned}F^2 &= g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\\ \Rightarrow\delta F^2 &= \delta g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}+g^{\mu\alpha}\delta g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\\ \Rightarrow\delta F^2 &= 2\delta g^{\mu\nu}F_{\mu\rho}F_\nu^\rho\end{aligned} \tag{3.17}$$

So, for m-form we should get,

$$\delta F_m^2 = mF_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m}\delta g^{\mu\nu} \tag{3.18}$$

Here, $\rho_2\ldots\rho_m$ are the remaining indices of the (m-1) summed over. One slot of F is occupied by the varied metric's index.

so, (3.14) becomes:

$$\begin{aligned}\delta S_{matter} &= -\int d^dx\frac{\epsilon}{2m!}\left(mF_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m}\delta g^{\mu\nu}\sqrt{|g|}-\frac{\sqrt{|g|}}{2}g_{\mu\nu}F_m^2\delta g^{\mu\nu}\right)\\ &= -\int d^dx\frac{\epsilon}{2m!}\sqrt{|g|}\left(mF_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m}-\frac{1}{2}g_{\mu\nu}F_m^2\right)\delta g^{\mu\nu}\\ &= -\int d^dx\frac{\epsilon}{2}\sqrt{|g|}\left(\frac{1}{(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m}-\frac{1}{2m!}g_{\mu\nu}F_m^2\right)\delta g^{\mu\nu}\end{aligned} \tag{3.19}$$

Now, comparing (3.19) with the matter action we know with energy momentum tensor,

$$T_{\mu\nu} = -\frac{1}{\sqrt{|g|}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \tag{3.20}$$

$$\Rightarrow\delta S_{matter} = -\sqrt{|g|}T_{\mu\nu}\delta g^{\mu\nu} \tag{3.21}$$

So, in light of (3.21) we get,

$$T_{\mu\nu} = \frac{\epsilon}{2}\left(\frac{1}{(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m}-\frac{1}{2m!}g_{\mu\nu}F_m^2\right) \tag{3.22}$$

Now the trace-reversed Einstein equations is given by:

$$R_{\mu\nu} = T_{\mu\nu}-\frac{1}{d-2}g_{\mu\nu}T \tag{3.23}$$

Taking the trace of (3.22),

$$
\begin{aligned}
g^{\mu\nu}T_{\mu\nu} &= T \\
&= g^{\mu\nu}\frac{\epsilon}{2}\left(\frac{1}{(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_{\nu}^{\rho_2\ldots\rho_m} - \frac{1}{2m!}g_{\mu\nu}F_m^2\right) \\
&= \frac{\epsilon}{2}\left(\frac{1}{(m-1)!}F_m^2 - \frac{d}{2m!}F_m^2\right) \\
&= \frac{\epsilon}{2}\left(\frac{1}{(m-1)!} - \frac{d}{2m!}\right) \\
&= \frac{\epsilon}{2}F_m^2\left(\frac{2m-d}{2m!}\right)
\end{aligned} \tag{3.24}
$$

Plugging into (3.23),

$$
\begin{aligned}
R_{\mu\nu} &= \epsilon\left(\frac{1}{2(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_{\nu}^{\rho_2\ldots\rho_m} - \frac{1}{4m!}g_{\mu\nu}F_m^2 - F_m^2 g_{\mu\nu}\frac{2m-d}{4m!(d-2)}\right) \\
&= \epsilon\left(\frac{1}{2(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_{\nu}^{\rho_2\ldots\rho_m} - g_{\mu\nu}F_m^2\frac{1}{m!}\left(\frac{1}{4} + \frac{2m-d}{4(d-2)}\right)\right) \\
&= \epsilon\left(\frac{1}{2(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_{\nu}^{\rho_2\ldots\rho_m} - g_{\mu\nu}F_m^2\frac{1}{m!}\left(\frac{m-1}{2(d-2)}\right)\right) \\
&= \epsilon\left(\frac{1}{2(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_{\nu}^{\rho_2\ldots\rho_m} - g_{\mu\nu}F_m^2\ \frac{m-1}{2m!(d-2)}\right)
\end{aligned} \tag{*}
$$

Equation 1.18

$$S = \int d^dx\sqrt{|g|}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right)$$

In forms notation:

$$\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) = (R\star 1) - \left(\frac{1}{2}d\phi\wedge\star d\phi\right) \tag{3.25}$$

Varying w.r.t ϕ:

$$
\begin{aligned}
\delta_\phi S &= -\int\delta\left(\frac{1}{2}d\phi\wedge\star d\phi\right) \\
&= -\frac{1}{2}\int(\delta d\phi\wedge\star d\phi + d\phi\wedge\star\delta d\phi) \\
&= -\int(\delta d\phi\wedge\star d\phi) \\
&= -\int_{\partial M} d(\delta\phi\wedge\star d\phi) - (-1)^p\int_M \delta\phi\wedge d(\star d\phi) \\
&= \int_M \delta\phi\wedge d(\star d\phi)
\end{aligned} \tag{3.26}
$$

We have used all the similar conditions used in the previous one and the fact that scalars are 0-forms. So we get,

$$d(\star d\phi) = 0 \tag{3.27}$$

and, $d^\dagger d\phi = \pm \star d \star d\phi$. So,

$$\begin{aligned} d^\dagger d\phi &= 0 \\ \Rightarrow \partial_\mu \partial^\mu \phi &= 0 \end{aligned} \tag{*}$$

Now, varying w.r.t the metric,

$$\delta_g S = \delta \int d^d x \sqrt{|g|} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right) \tag{3.28}$$

Which, similar to the previous one gives us two parts,

$$\delta_g S_{gravity} = \int d^d x \left(\delta \sqrt{|g|} R + \sqrt{|g|} \delta R \right) \tag{3.29}$$

$$\delta_g S_{scalar} = \int d^d x \ \delta \left(- \frac{\sqrt{|g|}}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) \tag{3.30}$$

Working one action at a time,

$$\begin{aligned} \delta_g S_{gravity} &= - \int d^d x \frac{1}{2} \sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu} R + \delta g^{\mu\nu} R_{\mu\nu} \sqrt{|g|} + g^{\mu\nu} \delta R_{\mu\nu} \sqrt{|g|} \\ &= - \int d^d x \frac{1}{2} \sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu} R + \delta g^{\mu\nu} R_{\mu\nu} \sqrt{|g|} \\ &= \int d^d x \sqrt{|g|} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} \end{aligned} \tag{3.31}$$

And now,

$$\begin{aligned} \delta_g S_{scalar} &= - \frac{1}{2} \int d^d x \left(\delta \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \delta g^{\mu\nu} \sqrt{|g|} \partial_\mu \phi \partial_\nu \phi \right) \\ &= - \frac{1}{2} \int d^d x \sqrt{|g|} \left(\delta g^{\mu\nu} - \frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} g^{\mu\nu} \right) \partial_\mu \phi \partial_\nu \phi \\ &= - \frac{1}{2} \int d^d x \sqrt{|g|} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\mu \phi \partial^\mu \phi \right) \delta g^{\mu\nu} \end{aligned} \tag{3.32}$$

Now, comparing again to (3.21),

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\mu \phi \partial^\mu \phi \right) \\ &= \left(\frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu\nu} \partial_\mu \phi \partial^\mu \phi \right) \end{aligned} \tag{3.33}$$

and similarly we take the trace to plug into the trace reversed form (3.23):

$$\begin{aligned}
g^{\mu\nu}T_{\mu\nu} &= T \\
&= \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g^{\mu\nu}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi \\
&= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}d\ \partial_\mu\phi\partial^\mu\phi
\end{aligned} \tag{3.34}$$

So we get,

$$\begin{aligned}
R_{\mu\nu} &= T_{\mu\nu} - \frac{1}{d-2}g_{\mu\nu}T \\
&= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}\ \partial_\mu\phi\partial^\mu\phi - \frac{1}{d-2}g_{\mu\nu}\left(\frac{1}{2} - \frac{d}{4}\right)\partial_\mu\phi\partial^\mu\phi \\
&= \left(\frac{1}{2} - \frac{1}{4} - \frac{1}{2(d-2)} + \frac{d}{4(d-2)}\right)\partial_\mu\phi\partial_\nu\phi \\
&= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi
\end{aligned} \tag{*}$$

Equation 2.26

$$S = \int d^dx\sqrt{|g|}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\epsilon}{2m!}e^{\beta\phi}F_m^2\right)$$

In forms notation:

$$\int\left(R\star 1 - \frac{1}{2}d\phi\wedge\star d\phi - \frac{\epsilon}{2}e^{\beta\phi}F_m\wedge\star F_m\right) \tag{3.35}$$

Varying w.r.t A_{m-1}, similar to how we got (3.10):

$$\begin{aligned}
\delta_{A_{m-1}}S &= -\frac{\epsilon}{2}\int e^{\beta\phi}(\delta F_m\wedge\star F_m) \\
&= -\epsilon\int\delta A_{m-1}\wedge d(e^{\beta\phi}\star F_m)
\end{aligned} \tag{3.36}$$

So, we have:

$$d(e^{\beta\phi}\star F_m) = 0 \tag{3.37}$$

$$\partial_\mu(\sqrt{|g|}e^{\beta\phi}F^{\mu\nu_1\ldots\nu_m}) = 0 \tag{3.38}$$

Now, varying w.r.t ϕ:

$$\delta_\phi S = \int\sqrt{|g|}\left(\frac{1}{2}\delta(d\phi\wedge\star d\phi) - \frac{\epsilon}{2}\beta\delta\phi e^{\beta\phi}F_m\wedge\star F_m\right) \tag{3.39}$$

Working on the inside,

$$
\begin{aligned}
&\frac{1}{2}\delta(d\phi \wedge \star d\phi) - \frac{\epsilon}{2}\beta\delta\phi e^{\beta\phi} F_m \wedge \star F_m \\
=&\delta\phi \wedge d(\star d\phi) - \frac{\epsilon}{2}\beta\delta\phi e^{\beta\phi} F_m \wedge \star F_m
\end{aligned} \tag{3.40}
$$

So, for $\delta S = 0$ we have,

$$
\begin{aligned}
d(\star d\phi) &= \frac{\epsilon\beta}{2} e^{\beta\phi} F_m \wedge \star F_m \\
\Rightarrow \partial_\mu \partial^\mu \phi &= \frac{\epsilon\beta}{2m!} F_{\mu_1 \ldots \mu_m} F^{\mu_1 \ldots \mu_m}
\end{aligned} \tag{3.41}
$$

Now, varying w.r.t the metric:

Taking the results from the trace reversed parts $R(F)_{\mu\nu}$(*) and $R(\phi)_{\mu\nu}$(*) with the exponential:

$$
\begin{aligned}
R_{\mu\nu} &= R(F)_{\mu\nu} + R(\phi)_{\mu\nu} \\
&= \epsilon e^{\beta\phi}\left(\frac{1}{2(m-1)!} F_{\mu\rho_2 \ldots \rho_m} F_\nu^{\rho_2 \ldots \rho_m} - g_{\mu\nu} F_m^2 \, \frac{m-1}{2m!(d-2)}\right) + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi \\
&= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon e^{\beta\phi}}{2(m-1)!}\left(F_{\mu\rho_2 \ldots \rho_m} F_\nu^{\rho_2 \ldots \rho_m} - \frac{m-1}{m(d-2)} g_{\mu\nu} F_m^2 \right)
\end{aligned} \tag{*}
$$

4 Reproducing Sabra's Solutions

The Kasner metric generalized:

$$
ds^2 = \epsilon_0 d\tau^2 + \sum_{i=1}^{d-1} \epsilon_i \tau^{2a_i} dx_i^2 \tag{4.1}
$$

with,

$$
\sum_{i=1}^{d-1} a_i = \sum_{i=1}^{d-1} a_i^2 = 1 \tag{4.2}
$$

$$
\begin{aligned}
&g_{\tau\tau=\epsilon_0} \\
&g_{ii} = \epsilon_i \tau^{2a_1}
\end{aligned} \tag{4.3}
$$

4.1 A d-dimensional Gravity Theory with an m-form

The action,

$$S = \int d^d x \sqrt{|g|} \left(R - \frac{\epsilon}{2m!} F_m^2 \right) \tag{4.4}$$

The equations of motion are:

$$R_{\mu\nu} - \epsilon \left(\frac{1}{2(m-1)!} F_{\mu\alpha_2 \ldots \alpha_m} F_\nu^{\alpha_2 \ldots \alpha_m} - g_{\mu\nu} \frac{(m-1)}{2m!(d-2) F_m^2} \right) = 0 \tag{4.5}$$

$$\partial_\mu (\sqrt{|g|} F^{\mu\nu_2 \ldots \nu_m}) = 0 \tag{4.6}$$

Now,

$$\partial_\tau g_{ii} = 2 a_i \epsilon_i \tau^{2a_i - 1} \tag{4.7}$$

The non-zero Christoffels:

$$\begin{aligned} \Gamma^\tau_{ii} &= \frac{1}{2} g^{\tau\tau} \partial_\tau g_{ii} \\ &= -\frac{1}{2\epsilon_0} (2 \epsilon_i a_i \tau^{2a_i - 1}) \\ &= -\frac{\epsilon_i a_i}{\epsilon_0} \tau^{2a_i - 1} \end{aligned} \tag{4.8}$$

$$\Gamma^i_{\tau i} = \frac{1}{2} g^{ii} \partial_\tau g_{ii} = \frac{a_i}{\tau} \tag{4.9}$$

From these we compute the Ricci tensors $R_{\tau\tau}$ and $R_{x_i x_i}$; We get,

$$R_{\tau\tau} = \frac{1}{\tau^2} \sum_{i=1}^{d-1} (a_i - a_i^2) \tag{Sabra Eqn 1.12 pt1}$$

$$R_{x_i x_i} = \epsilon_0 \epsilon_i \tau^{2a_i - 2} a_i \left(1 - \sum_{k=1}^{d-1} a_k \right) \tag{Sabra Eqn 1.12 pt2}$$

Now, considering the m-form:

$$F_m = P dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m \tag{4.10}$$

Which we get from,

$$F_m = \frac{1}{m!} F_{\mu_1 \mu_2 \ldots \mu_m} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_m} \tag{4.11}$$

and, since $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$ already includes $m!$, we have,

$$F_m = F_{12 \ldots m} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m = P dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m \tag{4.12}$$

where, $F_{12\dots m} = P$.

Now,

$$F^{12\dots m} = g^{11} g^{22\dots} g^{mm} F_{12\dots m} = (\epsilon_1 \epsilon_2 \dots \epsilon_m)^{-1} \tau^{-2(a_1+a_2+\dots+a_m)} P \tag{4.13}$$

Now, for F_m^2,

$$\begin{aligned} F_m^2 &= F_{\mu_1\mu_2\dots\mu_m} F^{\mu_1\mu_2\dots\mu_m} \\ &= m! F_{12\dots m} F^{12\dots m} \\ &= m! P^2 (\epsilon_1 \epsilon_2 \dots \epsilon_m)^{-1} \tau^{-2S_m} \end{aligned} \tag{4.14}$$

Now, going back to $\frac{1}{2(m-1)!} F_{\mu\alpha_2\dots\alpha_m} F_\nu^{\alpha_2\dots\alpha_m}$. We have,

$$\frac{1}{2(m-1)!} F_{\mu\alpha_2\dots\alpha_m} F_\nu^{\alpha_2\dots\alpha_m} = \frac{1}{2} P^2 (\epsilon_1 \epsilon_2 \dots \epsilon_m)^{-1} \epsilon_i \tau^{-2(S_m - a_i)} \tag{4.15}$$

[3]

Now, for τ,

$$R_{\tau\tau} - \epsilon \left(\frac{1}{2(m-1)!} F_{\tau\alpha_2\dots\alpha_m} F_\tau^{\alpha_2\dots\alpha_m} - g_{\tau\tau} \frac{(m-1)}{2m!(d-2)} F_m^2 \right)$$

$$= R_{\tau\tau} + g_{\tau\tau} \frac{(m-1)}{2m!(d-2)} F_m^2 \qquad \text{(First term in parenthesis is zero since no time dependence.)}$$

$$= \frac{1}{\tau^2} \sum_{i=1}^{d-1} (a_i - a_i^2) + \epsilon_0 \frac{(m-1)}{2m!(d-2)} P^2 (\epsilon_1 \epsilon_2 \dots \epsilon_m)^{-1} \tau^{-2S_m} = 0 \tag{4.16}$$

Now, in (Sabra Eqn 1.12 pt1), we can see that it scales as,

$$R_{\tau\tau} \propto \tau^{-2} \tag{4.17}$$

But for (4.14),

$$F_m^2 \propto \tau^{-2S_m} \tag{4.18}$$

So, in order for them to scale in the same way:

$$S_m = 1 \qquad \text{(Similar to Kasner constraint)} \tag{4.19}$$

[3] ϵ_i & $(S_m - a_i)$ comes from the lower index and (m-1)!

Now, for $R_{x_ix_i}$,

$$\epsilon_0\epsilon_i\tau^{2a_i-2}a_i\left(1-\sum_{i=1}^{d-1}a_i\right)-\frac{\epsilon}{2}P^2(\epsilon_1\epsilon_2\dots\epsilon_m)^{-1}\epsilon_i\tau^{-2(S_m-a_i)}$$
$$+g_{x_ix_i}\frac{(m-1)}{2m!(d-2)}\epsilon\ m!P^2(\epsilon_1\epsilon_2\dots\epsilon_m)^{-1}\tau^{-2S_m} \tag{4.20}$$

$$=\epsilon_0\epsilon_i\tau^{2a_i-2}a_i\left(1-\sum_{i=1}^{d-1}a_i\right)-\frac{\epsilon}{2}P^2(\epsilon_1\epsilon_2\dots\epsilon_m)^{-1}\epsilon_i\tau^{-2(S_m-a_i)}$$
$$+\sum_{i-1}^{d-1}\epsilon_1\tau^{2a_i}\frac{(m-1)}{2m!(d-2)}\epsilon\ m!P^2(\epsilon_1\epsilon_2\dots\epsilon_m)^{-1}\tau^{-2S_m}$$

$$=\epsilon_0\epsilon_i\tau^{2a_i-2}a_i\left(1-\sum_{i=1}^{d-1}a_i\right)-\frac{\epsilon}{2}P^2(\epsilon_1\epsilon_2\dots\epsilon_m)^{-1}\epsilon_i\tau^{-2(S_m-a_i)}$$
$$+\sum_{i=1}^{d-1}\epsilon_i\epsilon(\epsilon_1\epsilon_2\dots\epsilon_m)P^2\tau^{-2(S_m-a_i)}$$
$$=\epsilon_0\epsilon_i\tau^{2a_i-2}a_i\left(1-\sum_{i-1}^{d-1}a_i\right)-\epsilon\epsilon_i(\epsilon_1\epsilon_2\dots\epsilon_m)^{-1}\tau^{-2(S_m-a_i)}P^2\left(\frac{1}{2}-\frac{(m-1)}{2(d-2)}\right) \tag{4.21}$$

Now, we know,

$$\epsilon_0\epsilon_i\tau^{2a_i-2}a_i\left(1-\sum_{i-1}^{d-1}a_i\right)-\epsilon\epsilon_i(\epsilon_1\epsilon_2\dots\epsilon_m)^{-1}\tau^{-2(S_m-a_i)}P^2\left(\frac{1}{2}-\frac{(m-1)}{2(d-2)}\right)=0$$
$$\Rightarrow\epsilon_0\epsilon_i\tau^{2a_i-2}a_i\left(1-\sum_{i-1}^{d-1}a_i\right)=\epsilon\epsilon_i(\epsilon_1\epsilon_2\dots\epsilon_m)^{-1}\tau^{2(S_m-a_i)}P^2\left(\frac{1}{2}-\frac{(m-1)}{2(d-2)}\right) \tag{4.22}$$

Applying the constraint, $S_m=1$

$$\epsilon_0a_i\left(1-\sum_{i=1}^{d-1}a_i\right)=-\epsilon(\epsilon_1\epsilon_2\dots\epsilon_m)^{-1}P^2\left(\frac{1}{2}-\frac{(m-1)}{2(d-2)}\right.$$
$$\Rightarrow\epsilon_0a_i\left(1-\sum_{i=1}^{d-1}a_i\right)=-\epsilon(\epsilon_1\epsilon_2\dots\epsilon_m)P^2\left(\frac{d-m-1}{2(d-2)}\right) \tag{4.23}$$

We can see here that,

$$\epsilon_0 a_i\left(1-\sum_{i=1}^{d-1} a_i\right) = Constant \tag{4.24}$$

So, we get something like,

$$\epsilon_0 a_1\left(1-S\right) = C \tag{4.25}$$

$$\epsilon_0 a_2\left(1-S\right) = C \tag{4.26}$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\epsilon_0 a_m\left(1-S\right) = C \tag{4.27}$$

So, we have,

$$a_1 = a_2 = \cdots = a_m \tag{4.28}$$

Previously, we had $i \leq m$ but to find a consistent solution for ds^2, we must solve for every a_i, that is including those corresponding to directions not appearing in F_m. That way it guarantees the spacetime as a whole satisfies Einstein's equations.

Similar to $R_{x_i x_i}$ we find $R_{x_j x_j}$ but here, $F_m = 0 \;\; (j \geq m) : F_{j\alpha_2 \cdots = 0}$. We only have the trace term,

$$R_{x_j x_j} + \epsilon\frac{m-1}{2m!(d-2)} g_{jj} F_m^2 = 0 \tag{4.29}$$

$$\Rightarrow \epsilon_0 a_j (1-S) = C \tag{4.30}$$

So, we have,

$$a_{m+1} = a_{m+2} = \cdots = a_{d-1} \tag{4.31}$$

Now, solving for the exponents,

$$\begin{aligned} \epsilon_0 a_i(1-S) &= -\epsilon(\epsilon_1 \ldots \epsilon_m)^{-1} P^2 \left(\frac{d-m-1}{2(d-2)}\right) \\ \Rightarrow a_m &= \frac{\epsilon(\epsilon_1 \ldots \epsilon_m)^{-1} P^2}{\epsilon_0(1-S)}\left(\frac{d-m-1}{2(d-2)}\right) \end{aligned} \tag{4.32}$$

So for the ratio we have,

$$\frac{a_m}{a_{d-1}} = -\frac{d-m-1}{m-1} \tag{4.33}$$

The ratio says that it expands/contracts in the opposite directions.
From the constraint $S_m = \sum_{i=1}^{m} a_i = 1$, and all of m has the same exponents,

$$ma_m = 1 \;\Rightarrow\; a_m = \frac{1}{m} \qquad \text{(Sabra Eqn 1.14 pt2)}$$

Plugging (Sabra Eqn 1.14 pt2) into the (4.33):

$$\frac{\frac{1}{m}}{a_{d-1}} = -\frac{d-m-1}{m-1}$$
$$\Rightarrow a_{d-1} = -\frac{m-1}{m(d-m-1)} \qquad \text{(Sabra Eqn 1.14 pt3)}$$

The total sum of the exponents is given by the m number of a_m which lie inside the m-form and the remaining of the $d-1$ which is $(d-1)-m$,

$$\begin{aligned} S = \sum_{i=1}^{d-1} a_i &= ma_m + (d-m-1)a_{d-1} \\ &= m\cdot\frac{1}{m} - (d-m-1)\frac{m-1}{m(d-m-1)} \\ &= 1 - \left(1-\frac{1}{m}\right) \end{aligned} \tag{4.34}$$

So, we have,

$$1 - S = \left(1-\frac{1}{m}\right) \qquad \text{and,} \qquad S = \frac{1}{m} \tag{4.35}$$

Now, plugging (4.35) and (Sabra Eqn 1.14 pt3) into (4.29),

$$\begin{aligned} &\epsilon_0 a_{d-1}(1-S) = -\epsilon(\epsilon_1 \ldots \epsilon_m)^{-1}\frac{m-1}{2(d-2)}P^2 \\ =&\epsilon_0\left(-\frac{m-1}{m(d-m-1)}\right)\left(\frac{m-1}{m}\right) = -\epsilon(\epsilon_1 \ldots \epsilon_m)^{-1}\frac{m-1}{2(d-2)}P^2 \\ =&\epsilon_0\left(-\frac{m-1}{m^2(d-m-1)}\right) = \epsilon(\epsilon_1 \ldots \epsilon_m)^{-1}\frac{P^2}{2(d-2)} \\ =&\epsilon_0\frac{2(d-2)(m-1)}{m^2(d-m-1)}(\epsilon_1 \ldots \epsilon_m) = \epsilon P^2 \end{aligned} \tag{4.36}$$

Now, something to note, we had $\epsilon = \pm 1$ and so we should have $\frac{1}{\epsilon} = \epsilon$. And there for we have,

$$P^2 = \epsilon\epsilon_0(\epsilon_1 \ldots \epsilon_m)\frac{2(d-2)(m-1)}{m^2(d-m-1)} \qquad \text{(Sabra Eqn 1.14 pt1)}$$

4.1.1 For a 4D Maxwell Field

For a 4D Maxwell field we have d=4 and m=2. From (Sabra Eqn 1.14 pt2) and (Sabra Eqn 1.14 pt3) we get,

$$a_m = \frac{1}{2} \qquad\qquad a_{d-1} = -\frac{1}{2} \tag{4.37}$$

We have 1 time and 3 spatial dimensions and the number of exponents are given by: $(d-1) = 4-1 = 3$. So the spatial exponents are a_1, a_2, a_3.

So, the Kasner metric looks like,

$$ds^2 = -d\tau^2 + \tau^{2\alpha_1}dx_1^2 + \tau^{2\alpha_2}dx_2^2 + \tau^{2\alpha_3}dx_3^2 \tag{4.38}$$

where, $a_1 = a_2 = 1/2$ [4] and $a_3 = -1/2$.[5]

Now, the field strenght constraint,

$$\begin{aligned} P^2 &= \epsilon\epsilon_0(\epsilon_1 \dots \epsilon_m)\frac{2(m-1)(d-2)}{m^2(d-m-1)} \\ &= \epsilon\epsilon_0(\epsilon_1\epsilon_2)(1) \\ &= \epsilon\epsilon_0\epsilon_1\epsilon_2 \end{aligned} \tag{Sabra Eqn 1.15}$$

Using the signature (-,+,+,+),

$$\epsilon_0 = 1 \qquad\qquad \epsilon_1 = \epsilon_2 = +1 \tag{4.39}$$

So,

$$P^2 = -\epsilon \tag{4.40}$$

P^2 is the square of the field magnitude and has to be positive for a real magnetic field. So for that, $\epsilon = -1$.

4.2 A d-dimensional Gravity Theory with a Dynamical Scalar Field

The action,

$$S = \int d^dx\sqrt{|g|}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) \tag{4.41}$$

Considering the metric (4.1), we have

$$g^{\tau\tau} = \frac{1}{\epsilon_0} \qquad\qquad g^{ii} = \frac{1}{\epsilon_0\tau^{2a_i}} \tag{4.42}$$

[4] It is because $F_2 = Pdx^1 \wedge dx^2$, so a_1, a_2 scales in the same direction.

[5] This does not satisfy the vacuum Kasner constraints, it's modified by the field.

Previously, we the equations of motion for this action to be,

$$\partial_\mu \partial^\mu \phi = 0 \tag{4.43}$$

$$R_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu \phi = 0 \tag{4.44}$$

Using,

$$\partial_\mu \partial^\mu \phi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu \phi) = 0 \tag{4.45}$$

For, $\phi = \phi(\tau)$,

$$\begin{aligned} &\frac{1}{\sqrt{|g|}}\partial_\tau(\sqrt{|g|}g^{\tau\tau}\partial_\tau \phi(\tau)) = 0 \\ \Rightarrow &\frac{1}{\sqrt{|g|}}\partial_\tau\left(\sqrt{|g|}\frac{\dot{\phi}(\tau)}{\epsilon_0}\right) = 0 \\ \Rightarrow &\frac{C_0}{\epsilon_0}\partial_\mu(T^S\dot{\phi}(\tau)) = 0 \end{aligned} \tag{4.46}$$

where, we took $\sqrt{|g|} = CT^S$
We have,

$$\begin{aligned} &T^S\dot{\phi}(\tau) = Constant = d_1 \\ \Rightarrow &\dot{\phi}(\tau) = d_1 T^{-S} \\ \Rightarrow &\phi(\tau) = \int d_1 T^{-2} d\tau \\ \Rightarrow &\phi(\tau) = \frac{d_1}{1-S}\tau^{1-S} + d_2 \end{aligned} \tag{4.47}$$

But if we apply the Kasner constraint (4.35),

$$\phi(\tau) = d_1 \int \tau^{-1} d\tau = d_1 \log \tau + d_2 \qquad \text{(Sabra Eqn 1.19)}$$

Now,

$$R_{\tau\tau} = \frac{1}{2}(\partial_\tau \phi)^2 = \frac{1}{2}\dot{\phi}^2 \tag{4.48}$$

Plugging in $\dot{\phi} = d_1 \ \tau^{-}1$ and (Sabra Eqn 1.12 pt1),

$$\begin{aligned}
&\frac{1}{\tau^2}\sum_{i=1}^{d-1}(a_i - a_i^2) = \frac{1}{2}\left(\frac{d_1}{\tau}\right)^2 \\
\Rightarrow &\sum_{i=1}^{d-1} a_i - \sum_{i=1}^{d-1} a_i^2 - \frac{d_1}{2} = 0 \\
\Rightarrow &1 - \sum_{i=1}^{d-1} a_i^2 - \frac{d_1}{2} = 0 \\
\Rightarrow &\sum_{i=1}^{d-1} a_i^2 + \frac{d_1}{2} = \sum_{i=1}^{d-1} a_i = 1
\end{aligned} \qquad \text{(Sabra Eqn 1.20)}$$

(Kasner-dilaton constraint)

4.3 Brane Solutions

We start with d-dimensional gravity theory with an m-form.

The action,

$$ds^2 = e^{2U(\tau)}\left(\epsilon_0 d\tau^2 + \sum_{i=1}^{p} \epsilon_i \tau^{2a_i} dx_i^2\right) + e^{2V(\tau)}\left(\sum_{j=p+1}^{d-1} \epsilon_j \tau^{2a_j} dx_j^2\right) \tag{4.49}$$

Where, $\epsilon_0, \epsilon_i, \epsilon_j$ take values ± 1 and a_i, a_j are all constants. Clearly we have $d = p+q+1$. The non-zero components of the Ricci tensors are,

$$R_{\tau\tau} = -q\ddot{V} - p\ddot{U} - q\dot{V}(\dot{V} - \dot{U}) - \frac{1}{\tau}((s-l)\dot{U} + 2l\dot{V}) - \frac{1}{\tau^2}\sum_{k=1}^{d-1}(a_k^2 - a_k) \tag{4.50}$$

$$R_{x_i x_i} = -\epsilon_0 \epsilon_i \tau^{2a_j}\left[\ddot{U} - \frac{a_i}{\tau^2} + \left(\dot{U} + \frac{a_i}{\tau}\right)\left((p-1)\dot{U} + q\dot{V} + \frac{l+s}{\tau}\right)\right] \tag{4.51}$$

$$R_{x_j x_j} = -\epsilon_0 \epsilon_j e^{2V-2U} \tau^{2a_j}\left[\ddot{V} - \frac{a_j}{\tau^2} + \left(\dot{V} + \frac{a_j}{\tau}\right)\left(q\dot{V} + (p-1)\dot{U} + \frac{l+s}{\tau}\right)\right] \tag{4.52}$$

Where,

$$l = \sum_{j=p+1}^{d-1} a_j \quad , \quad s = \sum_{i=1}^{p} a_i \tag{4.53}$$

Imposing the relation introduced by Sabra in the paper,

$$qV + (p-1)U = 0 \tag{4.54}$$

We get the Ricci tensors to be,

$$R_{\tau\tau} = -\ddot{U} - \left[1 - 2\left(\frac{d-2}{q}\right)l\right]\frac{\dot{U}}{\tau} - \frac{(p-1)(d-2)}{q}\dot{U}^2 \tag{4.55}$$

$$R_{x_ix_i} = -\epsilon_0\epsilon_i\tau^{2a_i}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right) \tag{4.56}$$

$$R_{x_jx_j} = \frac{(p-1)}{q}\epsilon_0\epsilon_j e^{2(\frac{2-d}{q})U}\tau^{2a_j}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right) \tag{4.57}$$

The form in consideration is,

$$F_p = P dx_1 \wedge dx_2 \wedge \cdots \wedge dx_p \tag{4.58}$$

Also from (4.54) we can see that,

$$\begin{aligned} q\dot{V} + (p-1)\dot{U} &= 0 \\ \Rightarrow \dot{V} &= -\frac{p-1}{q}\dot{U} \\ \Rightarrow \ddot{V} &= -\left(\frac{p-1}{q}\right)\ddot{U} \end{aligned} \tag{4.59}$$

we can see here that V and U are not independent.
Moreover, for i we have,

$$F_{x_1\ldots x_p} = P \tag{4.60}$$

$$F^{x_1\ldots x_p} = P(\epsilon_1\ldots\epsilon_p)^{-1}e^{2(1-p)U}\tau^{2a_i-2s} \tag{4.61}$$

$$F_p^2 = p!P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{-2U}e^{2(1-p)U}\tau^{-2s} \tag{4.62}$$

From,

$$R_{\mu\nu} = \epsilon\left[\frac{1}{2(p-1)!}F_{\mu\alpha_2\ldots\alpha_p}F_\nu^{\alpha_2\ldots\alpha_p} - g_{\mu\nu}\frac{(p-1)}{2p!(d-2)}F_p^2\right] \tag{4.63}$$

The first part,

$$\begin{aligned} &\frac{1}{2(p-1)!}F_{i\alpha_2\ldots\alpha_p}F_i^{\alpha_2\ldots\alpha_p} \\ =&\frac{1}{2(p-1)!}(p-1)!\epsilon_i(\epsilon_1\ldots\epsilon_p)^{-1}P^2e^{2(1-p)U}\tau^{2a_i-2s} \end{aligned} \tag{4.64}$$

And the second part,

$$
\begin{aligned}
& - g_{ii}\frac{p-1}{2p!(d-2)}F_p^2 \\
=& - g_{ii}\frac{p-1}{2p!(d-2)}p!P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{2(1-p)U}\tau^{-2s}e^{-2U} \\
=& - \epsilon_i\tau^{2a_i}\frac{p-1}{2(d-2)}P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{2U}e^{2(1-p)U}e^{-2U}\tau^{-2s} \\
=& - \epsilon_i\frac{p-1}{2(d-2)}P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{2(1-p)U}\tau^{2a_i-2s}
\end{aligned} \tag{4.65}
$$

Combining them together,

$$
\begin{aligned}
R_{x_ix_i} &= \epsilon\epsilon_i\tau^{2a_i-2s}P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{2(1-p)U}\frac{q}{2(d-2)} \\
\Rightarrow - \epsilon\epsilon_0\tau^{2a_i}\left(\ddot{U}+\frac{\dot{U}}{\tau}\right) &= \epsilon\epsilon_i\tau^{2a_i-2s}P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{2(1-p)U}\frac{q}{2(d-2)} \\
\Rightarrow \ddot{U}+\frac{\dot{U}}{\tau}+\epsilon\epsilon_0(\epsilon_1\ldots\epsilon_p)^{-1}&\frac{q}{2(d-2)}P^2e^{2(1-p)U}\tau^{-2s}=0
\end{aligned} \tag{4.66}
$$

Now, for $R_{\tau\tau}$, the first part vanishes since the flux has no time dependence. So we have only the trace term,

$$
\begin{aligned}
& R_{\tau\tau} = -g_{\tau\tau}\epsilon\frac{p-1}{2(d-2)}P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{2(1-p)U}\tau^{-2s}e^{-2U} \\
& \Rightarrow R_{\tau\tau} = -\epsilon_0\frac{p-1}{2(d-2)}P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{2(1-p)U}\tau^{-2s} \\
& \Rightarrow - \ddot{U} - \left[1-2\left(\frac{d-2}{q}\right)l\right]\frac{\dot{U}}{\tau} - \frac{(p-1)(d-2)}{q}\dot{U}^2 = -\epsilon_0\frac{p-1}{2(d-2)}P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{2(1-p)U}\tau^{-2s} \\
& \Rightarrow - \ddot{U} - \alpha\frac{\dot{U}}{\tau} - \frac{(p-1)(d-2)}{q}\dot{U}^2 = -\epsilon_0\frac{p-1}{2(d-2)}P^2(\epsilon_1\ldots\epsilon_p)^{-1}e^{2(1-p)U}\tau^{-2s}
\end{aligned} \tag{4.67}
$$

Now,

$$
-\frac{(p-1)(d-2)}{q}\dot{U}^2 = \beta\dot{U}^2 \tag{4.68}
$$

We have the equations of motions to be,

$$
\ddot{U}+(1-2l)\frac{\dot{U}}{\tau}+(p-1)\dot{U}^2=0 \tag{4.69}
$$

$$
\ddot{U}+\frac{\dot{U}}{\tau}+\epsilon\epsilon_0\epsilon_1\ldots\epsilon_p\frac{q}{2(d-2)}e^{2(1-p)U}P^2\tau^{-2s}=0 \tag{4.70}
$$

Setting $f(\tau) = \dot{U}$,

$$\begin{aligned} &\ddot{U} + (1-2l)\frac{\dot{U}}{\tau} + (p-1)\dot{U}^2 = 0 \\ \Rightarrow &f'(\tau) + (1-2l)\frac{f(\tau)}{\tau} + (p-1)f(\tau)^2 = 0 \end{aligned} \tag{4.71}$$

Choosing,

$$Y(\tau) = e^{(p-1)U(\tau)} \tag{4.72}$$

Solving for $\dot{U}$,

$$\begin{aligned} &\ln Y(\tau) = (p-1)U(\tau) \\ \Rightarrow &\frac{\dot{Y}}{Y} = (p-1)\dot{U} \\ \Rightarrow &\dot{U} = \frac{1}{(p-1)}\frac{\dot{Y}}{Y} = f \\ \Rightarrow &f' = \frac{1}{(p-1)}\frac{\ddot{Y}Y - (\dot{Y})^2}{Y^2} \end{aligned} \tag{4.73}$$

Plugging (4.73) into (4.71),

$$\begin{aligned} &\frac{1}{(p-1)}\frac{\ddot{Y}Y - (\dot{Y})^2}{Y^2} + \frac{(1-2l)}{\tau(p-1)}\frac{\dot{Y}}{Y} + \left(\frac{\dot{Y}}{Y}\right)^2\frac{1}{(p-1)} = 0 \\ \Rightarrow &\ddot{Y}Y - (\dot{Y})^2 + \frac{1-2l}{\tau}Y\dot{Y} + (\dot{Y})^2 = 0 \\ \Rightarrow &\ddot{Y}Y + \frac{(1-2l)}{\tau}Y\dot{Y} = 0 \\ \Rightarrow &\ddot{Y} + \frac{(1-2l)}{\tau}\dot{Y} = 0 \end{aligned} \tag{4.74}$$

Substituting $Z = \dot{Y}$ and $Z' = \ddot{Y}$,

$$\frac{Z'}{Z} = -\frac{(1-2l)}{\tau}$$

$$\Rightarrow \frac{1}{Z} \cdot \frac{dZ}{d\tau} = \frac{(2l-1)}{\tau}$$

$$\Rightarrow \int \frac{dZ}{Z} = \int \frac{(2l-1)}{\tau} \cdot d\tau$$

$$\Rightarrow \ln Z = (2l-1) \ln \tau + \ln C$$

$$\Rightarrow Z = A\tau^{(2l-1)}$$

$$\Rightarrow \dot{Y} = A\tau^{2l-1}$$

$$\Rightarrow \frac{dY}{d\tau} = A\tau^{2l-1}$$

$$\Rightarrow \int dY = A \int \tau^{2l-1} d\tau$$

$$\Rightarrow Y = A\frac{\tau^2 l}{2l} + A_0$$

$$\Rightarrow Y = c_1 + c_2\tau^{2l}$$

$$\Rightarrow e^{(p-1)U(\tau)} = c_1 + c_2\tau^{2l}$$

$$\Rightarrow e^{U(\tau)} = (c_1 + c_2\tau^{2l})^{1/(p-1)}$$

$$\Rightarrow U(\tau) = \frac{1}{(p-1)} \ln (c_1 + c_2\tau^{2l}) \tag{4.75}$$

Working out each parts for (4.66),

$$\dot{U} = \frac{2l}{(p-1)} \frac{c_2 \tau^{2l-1}}{c_1 + c_2 \tau^{2l}} \tag{4.76}$$

$$\Rightarrow \ddot{U} = \frac{2l}{(p-1)} \frac{(2l-1) c_1 c_2 \tau^{2l-2} - c_2^2 \tau^{4l-2}}{(c_1 + c_2 \tau^{2l})^2} \tag{4.77}$$

We also have,

$$\frac{\dot{U}}{\tau} = \frac{2l}{(p-1)} \cdot \frac{c_2 \tau^{2l-2}}{c_1 + c_2 \tau^{2l}} \tag{4.78}$$

$$e^{2(1-p)U} = (c_1 + c_2 \tau^{2l})^{-2} \tag{4.79}$$

So, we have (4.66) to be,

$$-\frac{4l^2}{(p-1)} \frac{c_1 c_2 \tau^{2l-2}}{(c_1 + c_2 \tau^{2l})^2} + \epsilon \epsilon_0 (\epsilon_1 \ldots \epsilon_p)^{-1} \frac{q}{2(d-2)} P^2 \frac{1}{(c_1 + c_2 \tau^{2l})^2 \tau^{2s}} = 0$$

$$\Rightarrow -\frac{4l^2}{(p-1)} c_1 c_2 \tau^{2l-2} + \epsilon \epsilon_0 (\epsilon_1 \ldots \epsilon_p)^{-1} \frac{q}{2(d-2)} P^2 \tau^{-2s} = 0 \tag{4.80}$$

Using the Kasner constraint,

$$\sum_{k=1}^{d-1} a_k = \sum_{k=1}^{d-1} a_k^2 = 1 \tag{4.81}$$

$$or, \ (s + l = 1) \tag{4.82}$$

$$\Rightarrow l = 1 - s \tag{4.83}$$

We then have for (4.80),

$$\tau^{-2s} \left(\frac{4l^2}{(p-1)} c_1 c_2 + \epsilon \epsilon_0 (\epsilon_1 \ldots \epsilon_p)^{-1} \frac{q}{2(d-2)} P^2 \right) = 0$$

$$\Rightarrow \frac{2(d-2)}{q} \left(\frac{4l^2}{(p-1)} c_1 c_2 + \epsilon \epsilon_0 (\epsilon_1 \ldots \epsilon_p)^{-1} \frac{q}{2(d-2)} P^2 \right) = 0$$

$$\Rightarrow \frac{8(d-2) l^2}{q(p-1)} c_1 c_2 + \epsilon \epsilon_0 (\epsilon_1 \ldots \epsilon_p)^{-1} P^2 = 0 \tag{Sabra Eqn 2.11}$$

5 Considering a Metric and Flux Ansatz

Considering an ansatz for a metric in 4 dimensions,

$$\begin{aligned} ds^2 &= e^{\alpha(t)}\bigg(-dt^2 + e^{\gamma_1(t)}dx_1^2 + e^{\gamma_2(t)}dx_2^2 + e^{\gamma_3(t)}dx_3^2\bigg) \\ \Rightarrow ds^2 &= -e^{\alpha(t)}dt^2 + e^{\alpha(t)+\gamma_1(t)}dx_1^2 + e^{\alpha(t)+\gamma_2(t)}dx_2^2 + e^{\alpha(t)+\gamma_3(t)}dx_3^2 \\ \Rightarrow ds^2 &= -e^{\alpha(t)}dt^2 + e^{\alpha(t)}\sum_{i=1}^{3} e^{\gamma_i(t)}dx_i^2 \end{aligned} \tag{5.1}$$

With a scalar flux $\phi(x,t)$ along with the flux ansatz,

$$F_2 = f_1(x)q_1(t)dt \wedge dx_1 + f_2(x)q_2(t)dt \wedge dx_2 + f_3(x)q_3(t)dt \wedge dx_3 \tag{5.2}$$

And the action is,

$$S = \int d^4x\sqrt{|g|}\bigg(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\epsilon}{2m!}e^{\beta\phi}F_2^2\bigg) \tag{5.3}$$

We know,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \Rightarrow ds^2 = g_{00}(dx^0)^2 + 2g_{0i}dx^0dx^i + g_{ij}dx^idx^j \tag{5.4}$$

But the metric does not have any cross terms so we have,

$$g_{0i} = 0 \qquad\qquad g_{ij} = 0 \text{ (for i} \neq \text{j)} \tag{5.5}$$

So, we have the form:

$$ds^2 = e^{\alpha(t)}\bigg(-dt^2 + \sum_{i=1}^{3} e^{\gamma_i(t)}dx_i^2\bigg) \tag{5.6}$$

where,

$$g_{00} = -e^{\alpha(t)} \qquad\qquad g^{00} = -e^{-\alpha(t)} \tag{5.7}$$

$$g_{ii} = e^{\alpha(t)+\gamma_i(t)} \qquad\qquad g^{ii} = e^{-[\alpha(t)+\gamma_i(t)]} \tag{5.8}$$

5.1 Computing the Ricci Tensors

The Christoffel connection is:

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{ac}) \tag{5.9}$$

The non-zero Christoffels are:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}g^{00}(\partial_0 g_{00}) = \frac{1}{2}(-e^{-\alpha})(-\dot{\alpha}e^{\alpha}) = \frac{1}{2}\dot{\alpha} \\
\Gamma^0_{ii} &= \frac{1}{2}g^{00}(\partial_0 g_{ii}) = \frac{1}{2}e^{-\alpha}(\dot{\alpha} + \dot{\gamma}_i)e^{\alpha+\gamma_i} = \frac{1}{2}e^{\gamma_i}(\dot{\alpha} + \dot{\gamma}_i) \qquad (5.10) \\
\Gamma^i_{0i} &= \Gamma^i_{i0} = \frac{1}{2}g^{ii}(\partial_0 g_{ii}) = \frac{1}{2}(\dot{\alpha} + \dot{\gamma}_i) \qquad (5.11)
\end{aligned}$$

The Ricci tensor is given by:

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^m_{km} - \Gamma^k_{im}\Gamma^m_{jk} \tag{5.12}$$

So, for R_{00} and R_{ij} we have,

$$R_{00} = \partial_k \Gamma^k_{00} - \partial_0 \Gamma^k_{0k} + \Gamma^k_{00}\Gamma^m_{km} - \Gamma^k_{0m}\Gamma^m_{0k} \tag{5.13}$$

Working term by term:

$$\partial_k \Gamma^k_{00} = \partial_0 \Gamma^0_{00} = \frac{1}{2}\ddot{\alpha} \tag{5.14}$$

$$-\partial_0 \Gamma^k_{0k} = -\partial_0(\Gamma^0_{00} + \sum_i \Gamma^i_{0i}) = -2\ddot{\alpha} - \sum_i \frac{1}{2}\ddot{\gamma}_i \tag{5.15}$$

$$\Gamma^k_{00}\Gamma^m_{km} = \frac{1}{2}\dot{\alpha}(2\dot{\alpha} + \sum_i \frac{1}{2}\dot{\gamma}_i) = \dot{\alpha}^2 + \frac{1}{4}\dot{\alpha}\sum_i \dot{\gamma}_i \tag{5.16}$$

$$-\Gamma^k_{0m}\Gamma^m_{0k} = -(\frac{1}{2}\dot{\alpha})^2 - \sum_i(\frac{1}{2}(\dot{\alpha} + \dot{\gamma}_i)^2) = -\dot{\alpha}^2 - \frac{1}{2}\dot{\alpha}\sum_i \dot{\gamma}_i - \frac{1}{4}\sum_i \dot{\gamma}_i{}^2 \tag{5.17}$$

So we have,

$$\begin{aligned}
R_{00} &= \frac{1}{2}\ddot{\alpha} - 2\ddot{\alpha} - \sum_i \frac{1}{2}\ddot{\gamma}_i + \dot{\alpha}^2 + \frac{1}{4}\dot{\alpha}\sum_i \dot{\gamma}_i - \dot{\alpha}^2 - \frac{1}{2}\dot{\alpha}\sum_i \dot{\gamma}_i - \frac{1}{4}\sum_i \dot{\gamma}_i{}^2 \\
&= -\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i \ddot{\gamma}_i - \frac{1}{4}\dot{\alpha}\sum_i \dot{\gamma}_i - \frac{1}{4}\sum_i \dot{\gamma}_i{}^2 \qquad (5.18)
\end{aligned}$$

Similarly for R_{ii} we get,

$$
\begin{aligned}
R_{ii} &= \partial_k \Gamma^k_{ii} - \partial_i \Gamma^k_{ik} + \Gamma^k_{ii}\Gamma^m_{km} - \Gamma^k_{im}\Gamma^m_{ik} \\
&= \frac{1}{2}e^{\gamma_i}\left(\ddot{\alpha} + \ddot{\gamma}_i + \dot{\gamma}_i(\dot{\alpha} + \dot{\gamma})\right) - 0 + \left(\frac{1}{2}e^{\gamma_i}(\dot{\alpha} + \dot{\gamma}_i)\right)\left(2\dot{\alpha} + \frac{1}{2}\sum_k \dot{\gamma}_k\right) \\
&- 2\left(\frac{1}{2}e^{\gamma_i}(\dot{\alpha} + \dot{\gamma}_i)\right)\left(\frac{1}{2}(\dot{\alpha} + \dot{\gamma}_i)\right) \\
&= \frac{1}{2}e^{\gamma_i}\left(\ddot{\alpha} + \ddot{\gamma}_i + \dot{\alpha}^2 + \dot{\alpha}\dot{\gamma}_i + \frac{1}{2}\dot{\alpha}\sum_k \dot{\gamma}_k + \frac{1}{2}\dot{\gamma}_i\sum_k \dot{\gamma}_k\right) \qquad (5.19)
\end{aligned}
$$

So we have,

$$
R_{00} = -\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i \ddot{\gamma}_i - \frac{1}{4}\dot{\alpha}\sum_i \dot{\gamma}_i - \frac{1}{4}\sum_i \dot{\gamma}_i{}^2 \qquad (5.20)
$$

$$
R_{ii} = \frac{1}{2}e^{\gamma_i}\left(\ddot{\alpha} + \ddot{\gamma}_i + \dot{\alpha}^2 + \dot{\alpha}\dot{\gamma}_i + \frac{1}{2}\dot{\alpha}\sum_k \dot{\gamma}_k + \frac{1}{2}\dot{\gamma}_i\sum_k \dot{\gamma}_k\right) \qquad (5.21)
$$

$$
R_{ij} = 0 \quad \text{(no cross terms)} \qquad (5.22)
$$

5.2 Computing the Trace Reversed Einstein Equations

From the previous exercise we found the energy momentum tensors of to be,

$$
T(\phi)_{\mu\nu} = \left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi\right) \qquad (5.23)
$$

$$
T(F)_{\mu\nu} = \frac{\epsilon e^{\beta\phi}}{2}\left(\frac{1}{(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m} - \frac{1}{2m!}g_{\mu\nu}F_m^2\right) \qquad (5.24)
$$

So, we should have,

$$
T_{\mu\nu} = T(\phi)_{\mu\nu} + T(F)_{\mu\nu} \qquad (5.25)
$$

$$
T_{\mu\nu} = \left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi\right) + \frac{\epsilon e^{\beta\phi}}{2}\left(\frac{1}{(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m} - \frac{1}{2m!}g_{\mu\nu}F_m^2\right) \qquad (5.26)
$$

And the trace,

$$
\begin{aligned}
T &= g^{\mu\nu}T_{\mu\nu} \\
&= g^{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi\right) + \frac{\epsilon e^{\beta\phi}}{2}\left(\frac{1}{(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m} - \frac{1}{2m!}g_{\mu\nu}F_m^2\right) \\
&= \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}d\ \partial_\mu\phi\partial^\mu\phi\right) + \frac{\epsilon e^{\beta\phi}}{2}\left(\frac{1}{(m-1)!}F_m^2 - \frac{1}{2m!}d\ F_m^2\right) \\
&= \left(\frac{1}{2} - \frac{d}{4}\right)\partial_\mu\phi\partial^\mu\phi + \frac{\epsilon e^{\beta\phi}}{2}\left(\frac{1}{(m-1)!} - \frac{d}{2m!}\right)F_m^2 \\
&= -\frac{(d-2)}{4}\partial_\mu\phi\partial^\mu\phi + \frac{\epsilon e^{\beta\phi}}{2}F_m^2\left(\frac{2m-d}{2m!}\right)
\end{aligned} \tag{5.27}
$$

Plugging into the trace reversed einstein equation,

$$
R_{\mu\nu} = T_{\mu\nu} - \frac{1}{(d-2)}g_{\mu\nu}T
$$

for the $\left(\frac{1}{(d-2)}g_{\mu\nu}T\right)$ part,

$$
\begin{aligned}
& g_{\mu\nu}\frac{1}{(d-2)}\cdot\left[-\frac{(d-2)}{4}\partial_\mu\phi\partial^\mu\phi + \frac{\epsilon e^{\beta\phi}}{2}F_m^2\left(\frac{2m-d}{2m!}\right)\right] \\
=& g_{\mu\nu}\left[-\frac{1}{4}\partial_\mu\phi\partial^\mu\phi + \frac{\epsilon e^{\beta\phi}}{2}F_m^2\left(\frac{2m-d}{2m!(d-2)}\right)\right] \\
=& -\frac{1}{4}\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon e^{\beta\phi}}{2}g_{\mu\nu}F_m^2\left(\frac{2m-d}{2m!(d-2)}\right)
\end{aligned} \tag{5.28}
$$

So, we should have,

$$
\begin{aligned}
R_{\mu\nu} =& \left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi\right) + \frac{\epsilon e^{\beta\phi}}{2}\left(\frac{1}{(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m} - \frac{1}{2m!}g_{\mu\nu}F_m^2\right) \\
& -\frac{1}{4}\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon e^{\beta\phi}}{2}g_{\mu\nu}F_m^2\left(\frac{2m-d}{2m!(d-2)}\right) \\
=& \left(\frac{1}{2} - \frac{1}{4}\right)\partial_\mu\phi\partial_\nu\phi - \frac{d}{4}\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon e^{\beta\phi}}{2}\left[\frac{1}{(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m} - g_{\mu\nu}F_m^2\left(\frac{2m-d}{2m!(d-2)} - \frac{1}{2m!}\right)\right] \\
=& \left(\frac{1-d}{4}\right)\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon e^{\beta\phi}}{2}\left[\frac{1}{(m-1)!}F_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m} - g_{\mu\nu}F_m^2\left(\frac{m-d+1}{m!(d-2)}\right)\right] \\
=& \left(\frac{1-d}{4}\right)\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon e^{\beta\phi}}{2m!}\left[mF_{\mu\rho_2\ldots\rho_m}F_\nu^{\rho_2\ldots\rho_m} - g_{\mu\nu}F_m^2\left(\frac{m-d+1}{(d-2)}\right)\right]
\end{aligned} \tag{5.29}
$$

And from here for R_{00}, R_{ij} and R_{ii} we get,

$$
\begin{aligned}
R_{00} &= \left(\frac{1-d}{4}\right)\partial_0\phi\partial_0\phi + \frac{\epsilon e^{\beta\phi}}{2m!}\left[mF_{0\rho_2\ldots\rho_m}F_0^{\rho_2\ldots\rho_m} - g_{00}F_m^2\left(\frac{m-d+1}{(d-2)}\right)\right] \\
&= \left(\frac{1-d}{4}\right)\dot{\phi}^2 + \frac{\epsilon e^{\beta\phi}}{2m!}\left[mF_{0\rho_2\ldots\rho_m}F_0^{\rho_2\ldots\rho_m} + e^{\alpha(t)}F_m^2\left(\frac{m-d+1}{(d-2)}\right)\right]
\end{aligned} \tag{5.30}
$$

$$
\begin{aligned}
R_{ij} &= \left(\frac{1-d}{4}\right)\partial_i\phi\partial_j\phi + \frac{\epsilon e^{\beta\phi}}{2m!}\left[mF_{i\rho_2\ldots\rho_m}F_j\rho_2\ldots\rho_m - g_{ij}F_m^2\left(\frac{m-d+1}{(d-2)}\right)\right] \\
&= \left(\frac{1-d}{4}\right)\partial_i\phi\partial_j\phi + \frac{\epsilon e^{\beta\phi}}{2m!}\left(mF_{i\rho_2\ldots\rho_m}F_j^{\rho_2\ldots\rho_m}\right)
\end{aligned} \tag{5.31}
$$

$$
\begin{aligned}
R_{ii} &= \left(\frac{1-d}{4}\right)\partial_i\phi\partial_i\phi + \frac{\epsilon e^{\beta\phi}}{2m!}\left[mF_{i\rho_2\ldots\rho_m}F_i^{\rho_2\ldots\rho_m} - g_{ii}F_m^2\left(\frac{m-d+1}{(d-2)}\right)\right] \\
&= \left(\frac{1-d}{4}\right)(\partial_i\phi)^2 + \frac{\epsilon e^{\beta\phi}}{2m!}\left[mF_{i\rho_2\ldots\rho_m}F_i^{\rho_2\ldots\rho_m} - e^{\alpha(t)+\gamma_i(t)}F_m^2\left(\frac{m-d+1}{(d-2)}\right)\right]
\end{aligned} \tag{5.32}
$$

The metric,

$$
ds^2 = e^{\alpha(t)}\left(-dt^2 + e^{\gamma_1(t)}dx_1^2 + e^{\gamma_2(t)}dx_2^2 + e^{\gamma_3(t)}dx_3^2\right) \tag{5.33}
$$

Generalized,

$$
ds^2 = e^{\alpha}(t)\left(-dt^2 + \sum_{i=1}^{3} e^{\gamma_i(t)}dx_i^2\right) \tag{5.34}
$$

where,

$$
g_{tt} = -e^{\alpha(t)} \qquad\qquad g_{ii} = \sum_{i-1}^{3} e^{\alpha(t)+\gamma_i(t)} \tag{5.35}
$$

The equations of motion obtained:

$$
R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon e^{\beta\phi}}{2}\left[F_{\mu\alpha}F_\nu^\alpha - \frac{1}{4}F_2^2 g_{\mu\nu}\right] \tag{5.36}
$$

$$
\partial_\mu\left(\sqrt{|g|}e^{\beta\phi}F^{\mu\nu}\right) = 0 \tag{5.37}
$$

$$
\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\partial^\mu\phi\right) = \frac{\beta}{4}\epsilon e^{\beta\phi}F_2^2 \tag{5.38}
$$

We also had:

$$R_{00} = -\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i \ddot{\gamma}_i - \frac{1}{4}\dot{\alpha}\sum_i \dot{\gamma}_i - \frac{1}{4}\sum_i \dot{\gamma_i}^2 \tag{5.39}$$

$$R_{ii} = \frac{1}{2}e^{\gamma_i}\left(\ddot{\alpha} + \ddot{\gamma}_i + \dot{\alpha}^2 + \dot{\alpha}\dot{\gamma}_i + \frac{1}{2}\dot{\alpha}\sum_k \dot{\gamma}_k + \frac{1}{2}\dot{\gamma}_i\sum_k \dot{\gamma}_k\right) \tag{5.40}$$

$$R_{ij} = 0 \quad \text{(no cross terms)} \tag{5.41}$$

Now,

$$\begin{aligned} F_2^2 =& F_{\mu\nu}F^{\mu\nu} \\ =& 2\left(F_{0i}g^{00}g^{ii}F_{0i} + F_{ij}g^{ii}g^{jj}F_{ij}\right) \\ =& 2\left(g^{00}g^{ii}F_{0i}^2\right) \\ =& -2e^{-2\alpha(t)-\gamma_i(t)}F_{0i}^2 \end{aligned} \tag{5.42}$$

and,

$$F_{0\alpha}F_\nu^\alpha = \sum_{i=1}^3 g^{ii}F_{0i}^2 = \sum_{i=1}^3 e^{-\alpha(t)-\gamma_i(t)}F_{0i}^2 \tag{5.43}$$

Considering for $\mu, \nu = 0$,

$$R_{00} - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}\left[F_{0\alpha}F_0^\alpha - \frac{1}{4}F_2^2 g_{tt}\right] = 0 \tag{5.44}$$

We have,

$$\begin{aligned} & R_{00} - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}\left[g^{ii}F_{0i}^2 - \frac{1}{4}F_2^2 g_{00}\right] = 0 \\ \Rightarrow & R_{00} - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}\left[e^{-\alpha(t)-\gamma_i(t)}F_{0i}^2 + \frac{1}{4}F_2^2 e^{\alpha(t)}\right] = 0 \\ \Rightarrow & R_{00} - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}\left[e^{-\alpha(t)-\gamma_i(t)}F_{0i}^2 - \frac{1}{2}e^{-2\alpha(t)-\gamma_i(t)} \cdot e^{\alpha(t)}F_{0i}^2\right] \\ \Rightarrow & R_{00} - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}F_{0i}^2\left[e^{-\alpha(t)-\gamma_i(t)} - \frac{1}{2}e^{-\alpha(t)-\gamma_i(t)}\right] = 0 \\ \Rightarrow & R_{00} - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}F_{0i}^2\left[\frac{1}{2}e^{-\alpha(t)-\gamma_i(t)}\right] = 0 \\ \Rightarrow & -\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i \ddot{\gamma}_i - \frac{1}{4}\dot{\alpha}\sum_i \dot{\gamma}_i - \frac{1}{4}\sum_i \dot{\gamma_i}^2 - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}F_{0i}^2\left[\frac{1}{2}e^{-\alpha(t)-\gamma_i(t)}\right] = 0 \\ \Rightarrow & -\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i \ddot{\gamma}_i - \frac{1}{4}\dot{\alpha}\sum_i \dot{\gamma}_i - \frac{1}{4}\sum_i \dot{\gamma_i}^2 - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}F_{0i}^2\left[\frac{1}{2}e^{-\alpha(t)-\gamma_i(t)}\right] = 0 \end{aligned} \tag{5.45}$$

Now, using the E.O.M (5.37),

$$\partial_\mu\left(\sqrt{|g|}e^{\beta\phi}F^{\mu\nu}\right) = 0$$
$$\Rightarrow\sqrt{|g|}e^{\beta\phi}F^{\mu\nu} = Constant \tag{5.46}$$

For (00) component we have,

$$e^{\beta\phi}g^{00}g^{ii}F_{0i} = constant$$
$$\Rightarrow - e^{\beta\phi}e^{-2\alpha(t)-\gamma_i(t)}F_{0i} = constant$$
$$\Rightarrow - e^{\beta\phi-2\alpha(t)-\gamma_i(t)}F_{0i} = constant \tag{5.47}$$

and for the determinant,

$$g_{\mu\nu} = diag(-e^{\alpha}, e^{\alpha+\gamma_1}, e^{\alpha+\gamma_2}, e^{\alpha+\gamma_3}) \tag{5.48}$$

So we have,

$$det(g) = (-e^{\alpha})(e^{\alpha+\gamma_1})(e^{\alpha+\gamma_2})(e^{\alpha+\gamma_3}) = -e^{4\alpha+\sum_1^3\gamma_i}$$
$$\Rightarrow\sqrt{|g|} = e^{2\alpha+\frac{1}{2}\sum_1^3\gamma_i} \tag{5.49}$$

Putting everything together,

$$e^{2\alpha+\frac{1}{2}\sum_1^3\gamma_q}(-e^{\beta\phi-2\alpha(t)-\gamma_i(t)}F_{0i}) = C$$
$$\Rightarrow - e^{\beta\phi-\gamma_i(t)}\cdot e^{\frac{1}{2}\gamma_q}F_{0i} = C$$
$$\Rightarrow F_{0i} = -Ce^{-\beta\phi}\cdot e^{-\frac{1}{2}\gamma_q-\gamma_i(t)} \tag{5.50}$$

Plugging (5.50) into (5.45),

$$-\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i\ddot{\gamma_i} - \frac{1}{4}\dot{\alpha}\sum_i\dot{\gamma_i} - \frac{1}{4}\sum_i\dot{\gamma_i}^2 - \frac{1}{2}\dot{\phi}^2 - C = 0$$
$$\Rightarrow -\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i\ddot{\gamma_i} - \frac{1}{4}\dot{\alpha}\sum_i\dot{\gamma_i} - \frac{1}{4}\sum_i\dot{\gamma_i}^2 - \frac{1}{2}\dot{\phi}^2 = C \tag{5.51}$$

From (5.38),

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\partial^\mu\phi\right) = \frac{\beta}{4}\epsilon e^{\beta\phi}F_2^2 \tag{5.52}$$

Where, $\sqrt{|g|} = e^{2\alpha(t)+\frac{1}{2}\sum^3 \gamma_i}$ and (5.42)

So, we have,

$$\begin{aligned}
\frac{1}{\sqrt{|g|}}\partial_\mu\Big(\sqrt{|g|}\partial^\mu\phi\Big) &= -\frac{\beta}{2}\epsilon e^{\beta\phi-2\alpha(t)-\gamma_i(t)}F_{0i}^2 \\
&= -\frac{\beta}{2}\epsilon e^{\beta\phi-2\alpha(t)-\gamma_i(t)}\Big(-Ce^{-\beta\phi-\frac{3}{2}\gamma_i(t)}\Big)^2 \\
&= -\frac{\beta}{2}\epsilon e^{-\beta\phi-2\alpha(t)-4\gamma_i(t)}C^2 \\
&= -\frac{C^2\beta}{2}\epsilon e^{-\beta\phi-2\alpha(t)-4\gamma_i(t)}
\end{aligned} \tag{5.53}$$

So,

$$\begin{aligned}
&\frac{1}{\sqrt{|g|}}\partial_\mu\Big(\sqrt{|g|}\partial^\mu\phi\Big) = -\frac{C^2\beta}{2}\epsilon e^{-\beta\phi-2\alpha(t)-4\gamma_i(t)} \\
\Rightarrow&\partial_\mu\Big(\sqrt{|g|}\partial^\mu\phi\Big) = -\sqrt{|g|}\frac{C^2\beta}{2}\epsilon e^{-\beta\phi-2\alpha(t)-4\gamma_i(t)} \\
\Rightarrow&(\partial_\mu\sqrt{|g|})\partial^\mu\phi + \sqrt{|g|}\partial_\mu\partial^\mu\phi = -e^{2\alpha(t)+\frac{1}{2}\gamma_i(t)-\beta\phi-2\alpha(t)-4\gamma_i(t)}\frac{C^2\beta}{2} \\
\Rightarrow&(\partial_\mu\sqrt{|g|})\partial^\mu\phi + \sqrt{|g|}\partial_\mu\partial^\mu\phi = -\frac{C^2\beta}{2}\epsilon e^{-\beta\phi-\frac{7}{2}\gamma_i(t)}
\end{aligned} \tag{5.54}$$

For 00,

$$\Big(\partial_t\sqrt{|g|}g^{tt}\partial_t\phi + \sqrt{|g|}g^{tt}\partial_t^2\phi\Big) = \frac{C^2\beta}{2}\epsilon e^{-\beta\phi-\frac{7}{2}\gamma_i(t)} \tag{5.55}$$

Attempting part by part,

$$\begin{aligned}
\partial_t\sqrt{|g|}g^{tt}\partial_t\phi =&(\partial_t e^{2\alpha(t)+\frac{1}{2}\sum^3\gamma_i})e^{-\alpha(t)}\dot{\phi} \\
=&\Big(2\dot{\alpha}(t)+\frac{1}{2}\sum^3\dot{\gamma_i}\Big)e^{2\alpha(t)+\frac{1}{2}\sum^3\gamma_i(t)}e^{-\alpha(t)}\dot{\phi} \\
=&\Big(2\dot{\alpha}(t)+\frac{1}{2}\sum^3\dot{\gamma_i}\Big)e^{\alpha(t)+\frac{1}{2}\sum^3\gamma_i(t)}\dot{\phi}
\end{aligned} \tag{5.56}$$

$$\sqrt{|g|}g^{tt}\ddot{\phi} = e^{\alpha(t)+\frac{1}{2}\sum^3\gamma_i}\ddot{\phi} \tag{5.57}$$

$$
\begin{aligned}
&\left(\partial_t\sqrt{|g|}g^{tt}\partial_t\phi+\sqrt{|g|}g^{tt}\partial_t^2\phi\right)\\
=&\left(2\dot{\alpha}(t)+\frac{1}{2}\sum^3\dot{\gamma}_i\right)e^{\alpha(t)+\frac{1}{2}\sum^3\gamma_i(t)}\dot{\phi}+e^{\alpha(t)+\frac{1}{2}\sum^3\gamma_i}\ddot{\phi}\\
=&\left[\left(2\dot{\alpha}(t)+\frac{1}{2}\sum^3\dot{\gamma}_i\right)\dot{\phi}+\ddot{\phi}\right]e^{\alpha(t)+\frac{1}{2}\sum^3\gamma_i(t)}
\end{aligned}
\tag{5.58}
$$

Putting everything together,

$$
\begin{aligned}
&\left[\left(2\dot{\alpha}(t)+\frac{1}{2}\sum^3\dot{\gamma}_i(t)\right)\right]e^{\alpha(t)+\frac{1}{2}\sum^3\gamma_i(t)}=-\frac{C^2\beta}{2}\epsilon e^{-\beta\phi-\frac{7}{2}\sum^3\gamma_i(t)}\\
\Rightarrow&\left[\left(2\dot{\alpha}(t)+\frac{1}{2}\sum^3\dot{\gamma}_i(t)\right)\right]=-\frac{C^2\beta}{2}\epsilon e^{-\alpha(t)-\beta\phi-4\gamma_i(t)}\\
\Rightarrow&\left[\left(2\dot{\alpha}(t)+\frac{1}{2}\sum^3\dot{\gamma}_i(t)\right)\right]+\frac{C^2\beta}{2}\epsilon e^{-\alpha(t)-\beta\phi-4\gamma_i(t)}=0
\end{aligned}
\tag{5.59}
$$

Substituting,

$$
\ddot{\phi}+P\dot{\phi}+Q=0\Rightarrow\dot{Z}+PZ+Q=0 \tag{5.60}
$$

Where,

$$
P=2\dot{\alpha}(t)+\frac{1}{2}\sum^3\dot{\gamma}_i \tag{5.61}
$$

$$
Q=\frac{C^2\beta}{2}\epsilon e^{-\alpha(t)-\beta\phi-4\gamma_i(t)} \tag{5.62}
$$

Using the integrating factor,

$$
\mu\dot{Z}+\mu PZ=-\mu Q \tag{5.63}
$$

Say, $\dot{\mu}=\mu P$,

$$
\begin{aligned}
&\dot{\mu}=\mu P\\
\Rightarrow&\frac{\dot{\mu}}{\mu}=P\\
\Rightarrow&\int\frac{\dot{\mu}}{\mu}dt=\int Pdt\\
\Rightarrow&\ln\mu=\int Pdt\\
\Rightarrow&\mu=e^{\int Pdt}
\end{aligned}
\tag{5.64}
$$

Now,

$$\frac{d}{dt}(\mu Z) = \mu\dot{Z} + \dot{\mu}Z = \mu\dot{Z} + \mu PZ \tag{5.65}$$

So we have,

$$\begin{aligned}
&\frac{d}{dt}(\mu Z) = -\mu Q \\
\Rightarrow &\mu Z = -\int \mu Q dt + C \\
\Rightarrow &Z = -\frac{1}{\mu}\int \mu Q dt + C_0
\end{aligned} \tag{5.66}$$

So we have,

$$\begin{aligned}
&\dot{\phi} = -\frac{1}{\mu}\int \mu Q dt + C_0 \\
\Rightarrow &\dot{\phi} = -e^{-\int P dt}\int e^{\int P dt}\left(\frac{C^2\beta}{2}\epsilon e^{-\alpha(t)-\beta\phi-4\gamma_i(t)}\right)dt + C_0 \\
\Rightarrow &\dot{\phi} = -\frac{C^2\beta}{2}e^{-\int P dt}\int e^{\int P dt-\alpha(t)-\beta\phi-4\gamma_i(t)}dt + C_0
\end{aligned} \tag{5.67}$$

Now,

$$\begin{aligned}
\int P dt &= \int \left(2\dot{\alpha}(t) + \frac{1}{2}\sum^{3}\dot{\gamma}_i(t)\right)dt \\
&= 2\alpha(t) + \frac{1}{2}\gamma_i(t) = \mu
\end{aligned} \tag{5.68}$$

So, we have,

$$\begin{aligned}
&\dot{\phi} = -\frac{C^2\beta}{2}e^{-2\alpha(t)-\frac{1}{2}\gamma_i(t)}\int e^{2\alpha+\frac{1}{2}\gamma_i(t)-\alpha(t)-\beta\phi-4\gamma_i(t)}dt + C_0 \\
\Rightarrow &\dot{\phi} = -\frac{C^2\beta}{2}e^{-\beta\phi-4\gamma_i(t)-\alpha(t)}
\end{aligned} \tag{5.69}$$

Going back to (5.45),

$$\begin{aligned}
&-\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i\ddot{\gamma_i} - \frac{1}{4}\dot{\alpha}\sum_i\dot{\gamma_i} - \frac{1}{4}\sum_i\dot{\gamma_i}^2 - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}F_{0i}^2\left[\frac{1}{2}e^{-\alpha(t)-\gamma_i(t)}\right] = 0 \\
\Rightarrow &-\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i\ddot{\gamma_i} - \frac{1}{4}\dot{\alpha}\sum_i\dot{\gamma_i} - \frac{1}{4}\sum_i\dot{\gamma_i}^2 - \frac{1}{2}\dot{\phi}^2 - \frac{\epsilon e^{\beta\phi}}{2}\left(-Ce^{-\beta\phi-\frac{3}{2}\gamma_i(t)}\right)^2\left[\frac{1}{2}e^{-\alpha(t)-\gamma_i(t)}\right] = 0 \\
\Rightarrow &-\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i\ddot{\gamma_i} - \frac{1}{4}\dot{\alpha}\sum_i\dot{\gamma_i} - \frac{1}{4}\sum_i\dot{\gamma_i}^2 - \frac{1}{2}\dot{\phi}^2 - \frac{C^2\epsilon}{4}e^{-\beta\phi-\alpha(t)-\gamma_i(t)} = 0
\end{aligned} \tag{5.70}$$

Plugging in (5.69),

$$-\frac{3}{2}\ddot{\alpha}-\frac{1}{2}\sum_i\ddot{\gamma}_i-\frac{1}{4}\dot{\alpha}\sum_i\dot{\gamma}_i-\frac{1}{4}\sum_i\dot{\gamma_i}^2-\frac{1}{2}\bigg(-\frac{C^2\beta}{2}e^{-\beta\phi-4\gamma_i(t)-\alpha(t)}\bigg)-\frac{C^2\epsilon}{4}e^{-\beta\phi-\alpha(t)-\gamma_i(t)}=0$$

$$\Rightarrow-\frac{3}{2}\ddot{\alpha}-\frac{1}{2}\sum_i\ddot{\gamma}_i-\frac{1}{4}\dot{\alpha}\sum_i\dot{\gamma}_i-\frac{1}{4}\sum_i\dot{\gamma_i}^2+\frac{C^2\beta}{4}e^{-\beta\phi-4\gamma_i(t)-\alpha(t)}-\frac{C^2\epsilon}{4}e^{-\beta\phi-\alpha(t)-\gamma_i(t)}=0 \tag{5.71}$$

Considering for $\mu,\nu=i$,

$$R_{ii}-\frac{1}{2}\partial_i\phi\partial_i\phi-\frac{\epsilon e^{\beta\phi}}{2}\bigg[F_{i\alpha}{F_i}^{\alpha}-\frac{1}{4}F_2^2g_{ii}\bigg]=0 \tag{5.72}$$

Now,

$$\begin{aligned}F_{i\alpha}F_i^{\alpha}=&\sum g^{00}F_{0i}^2\\ =&e^{-\alpha(t)}F_{0i}^2\\ =&-e^{-\alpha(t)}\bigg(-Ce^{-\beta\phi-\frac{3}{2}\gamma_i(t)}\bigg)^2\\ =&-C^2e^{-\alpha(t)-2\beta\phi-3\gamma_i}\end{aligned} \tag{5.73}$$

and,

$$\begin{aligned}&\frac{1}{4}F_2^2g_{ii}\\ =&\frac{1}{4}(-2e^{-2\alpha(t)-\gamma_i(t)}F_{0i}^2)\cdot g_{ii}\\ =&\frac{1}{4}\bigg[-2e^{-2\alpha(t)-\gamma_i(t)}\bigg(-Ce^{-\beta\phi-\frac{3}{2}\gamma_i(t)}\bigg)^2\bigg]\cdot g_{ii}\\ =&-\frac{C^2}{2}e^{-2\alpha(t)-2\beta\phi-4\gamma_i(t)}\cdot e^{\alpha(t)+\frac{1}{2}\gamma_i(t)}\\ =&-\frac{C^2}{2}e^{-\alpha(t)-2\beta\phi-\frac{7}{2}\gamma_i(t)}\end{aligned} \tag{5.74}$$

So, we have,

$$\begin{aligned}&R_{ii}-\frac{1}{2}\partial_i^2\phi-\frac{\epsilon e^{\beta\phi}}{2}\bigg[-C^2e^{-\alpha(t)-2\beta\phi-3\gamma_i}+\frac{C^2}{2}e^{-\alpha(t)-2\beta\phi-\frac{7}{2}\gamma_i(t)}\bigg]=0\\ \Rightarrow&R_{ii}-\frac{1}{2}\partial_i^2\phi-\frac{\epsilon}{2}\bigg[-C^2e^{-\alpha(t)-\beta\phi-3\gamma_i}+\frac{C^2}{2}e^{-\alpha(t)-\beta\phi-\frac{7}{2}\gamma_i(t)}\bigg]=0\\ \Rightarrow&R_{ii}-\frac{1}{2}\partial_i^2\phi-\frac{C^2\epsilon}{2}e^{-\alpha(t)-\beta\phi}\bigg[-e^{-3\gamma_i}+\frac{1}{2}e^{-\frac{7}{2}\gamma_i(t)}\bigg]=0\end{aligned} \tag{5.75}$$

Again, from (5.38),

$$
\begin{aligned}
&\frac{1}{\sqrt{|g|}}\partial_i\Bigg(\sqrt{|g|}\partial^i\phi\Bigg) = \frac{\beta}{4}\epsilon e^{\beta\phi}F_2^2 \\
\Rightarrow&\partial_i\Bigg(\sqrt{|g|}\partial^i\phi\Bigg) = \sqrt{|g|}\frac{\beta\epsilon}{4}e^{\beta\phi}F_2^2 \\
\Rightarrow&(\partial_i\sqrt{|g|})\partial^i\phi + \sqrt{|g|}\partial_i\partial^i\phi = \sqrt{|g|}\frac{\beta\epsilon}{4}e^{\beta\phi}F_2^2 \\
\Rightarrow&\sqrt{|g|}\partial_i\partial^i\phi = \sqrt{|g|}\frac{\beta\epsilon}{4}e^{\beta\phi}F_2^2 - (\partial_i\sqrt{|g|})\partial^i\phi \\
\Rightarrow&\partial_i\partial^i\phi = \frac{\beta\epsilon}{4}e^{\beta\phi}F_2^2 - \frac{1}{\sqrt{|g|}}(\partial_i\sqrt{|g|})\partial^i\phi \\
\Rightarrow&g^{ii}\partial_i^2\phi = \frac{\beta\epsilon}{4}e^{\beta\phi}F_2^2 - \frac{1}{\sqrt{|g|}}(\partial_i\sqrt{|g|})\partial^i\phi \\
\Rightarrow&e^{-\alpha(t)-\gamma_i(t)}\partial_i^2\phi = \frac{\beta\epsilon}{4}e^{\beta\phi}F_2^2 - \frac{1}{\sqrt{|g|}}(\partial_i\sqrt{|g|})\partial^i\phi \\
\Rightarrow&\partial_i^2\phi = \frac{\beta\epsilon}{4}e^{\beta\phi+\alpha(t)+\gamma_i(t)}F_2^2 - \frac{1}{\sqrt{|g|}}(\partial_i\sqrt{|g|})\partial^i\phi
\end{aligned} \tag{5.76}
$$

Solving part by part,

$$
\begin{aligned}
&\frac{\beta\epsilon}{4}e^{\beta+\alpha(t)+\gamma_i(t)}F_2^2 \\
&= -\frac{\beta\epsilon}{2}e^{\beta\phi+\alpha(t)+\gamma_i(t)}\cdot e^{-2\alpha(t)-\gamma_i(t)}\Bigg(F_{ij}F^{ij}\Bigg) \\
&= 0
\end{aligned} \tag{5.77}
$$

and,

$$
\begin{aligned}
&\frac{1}{\sqrt{|g|}}(\partial_i\sqrt{|g|})g^{ii}\partial_i\phi \\
=&e^{-2\alpha(t)-\frac{1}{2}\gamma_i(t)}\Bigg(\partial_i(e^{2\alpha(t)+\frac{1}{2}\gamma_i(t)})\Bigg)e^{-\alpha(t)-\gamma_i(t)}\partial_i\phi \\
=&0
\end{aligned} \tag{5.78}
$$

So, we have,

$$
\partial_i^2\phi = 0 \tag{5.79}
$$

Now, plugging this into (5.75),

$$
\begin{aligned}
&R_{ii} - \frac{C^2\epsilon}{2}e^{-\alpha(t)-\beta\phi}\Big[-e^{-3\gamma_i} + \frac{1}{2}e^{-\frac{7}{2}\gamma_i(t)}\Big] = 0 \\
\Rightarrow &\frac{1}{2}e^{\gamma_i}\Big(\ddot{\alpha} + \ddot{\gamma}_i + \dot{\alpha}^2 + \dot{\alpha}\dot{\gamma}_i + \frac{1}{2}\dot{\alpha}\sum_k \dot{\gamma}_k + \frac{1}{2}\dot{\gamma}_i\sum_k \dot{\gamma}_k\Big) - \frac{C^2\epsilon}{2}e^{-\alpha(t)-\beta\phi}\Big[-e^{-3\gamma_i} + \frac{1}{2}e^{-\frac{7}{2}\gamma_i(t)}\Big] = 0
\end{aligned} \tag{5.80}
$$

Now for $\mu = i$ and $\nu = j$,

$$
\begin{aligned}
&R_{ij} - \frac{1}{2}\partial_i\phi\partial_j\phi - \frac{\epsilon e^{\beta\phi}}{2}\Big[F_{i\alpha}F_j^{\alpha} - \frac{1}{4}F_2^2 g_{ij}\Big] = 0 \\
\Rightarrow &0 - \frac{1}{2}\partial_i\phi\partial_j\phi + \frac{\epsilon e^{\beta\phi}}{8}F_2^2 g_{ij} = 0 \\
\Rightarrow &\frac{1}{2}\partial_i\phi\partial_j\phi = 0 \\
\Rightarrow &\partial_i\phi \cdot \partial_j\phi = 0
\end{aligned} \tag{5.81}
$$

So,

$$
\phi = Constant \tag{5.82}
$$

For $\mu, \nu = 0$,

$$
-\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i \ddot{\gamma}_i - \frac{1}{4}\dot{\alpha}\sum_i \dot{\gamma}_i - \frac{1}{4}\sum_i \dot{\gamma_i}^2 + \frac{C^2\beta}{4}e^{-\beta\phi-4\gamma_i(t)-\alpha(t)} - \frac{C^2\epsilon}{4}e^{-\beta\phi-\alpha(t)-\gamma_i(t)} = 0 \tag{5.83}
$$

For $\mu, \nu = i$,

$$
\frac{1}{2}e^{\gamma_i}\Big(\ddot{\alpha} + \ddot{\gamma}_i + \dot{\alpha}^2 + \dot{\alpha}\dot{\gamma}_i + \frac{1}{2}\dot{\alpha}\sum_k \dot{\gamma}_k + \frac{1}{2}\dot{\gamma}_i\sum_k \dot{\gamma}_k\Big) - \frac{C^2\epsilon}{2}e^{-\alpha(t)-\beta\phi}\Big[-e^{-3\gamma_i} + \frac{1}{2}e^{-\frac{7}{2}\gamma_i(t)}\Big] = 0 \tag{5.84}
$$

6 Conclusion

While the expectation of going beyond Kasner solutions with stringy ingredients; form fields, the dilation coupled to the form field, making it a fitting candidate for toy models of string theory, without having the final solutions still leaves it inconclusive. However, the results so far seems quite promising. We have a scalar field which varies in the time component and to be considered a rolling scalar field, requires a potential $V(\phi)$. But, the background solution here has no potential so rolling in not in the picture. Interestingly, we have the results so far while having the Kasner exponents being time dependent. Coming from the metric ansatz (5.1), we calculated the Ricci tensors, the equations of motion and acquiring the trace reversed Einstein equations. From the equations of motion we got that the scalar field has evolution in the time components. The 2-form flux ansatz took the calculations further to a suitable halt for the time being with expectations of this being a good candidate for Kination epoch, early universe anisotropy, String theory and Supergravity, if it also satisfies $R < 0$ and has rolling from fluctuations around the background. Moreover, questions may arise from a holographic point of view (AdS/CFT, which also has the condition for $R < 0$ and constant). In standard AdS/CFT we have asymptotically Anti-de Sitter space but for the Kasner metric we have expansion or contraction (non static) in time with anisotropy. So, the question arises that can one define a dual quantum theory in a universe with the boundary being time-dependent? A more concrete path can be found as we keep progressing with the calculations and properly evaluate the results.

Appendix

A Differential Forms

A.1 Types of Forms

0-Forms:

A 0-form is just a function:

$$f : \mathbb{R} \to \mathbb{R}$$

1-Forms:

A 1-form is a liner combination of differentials:

$$\omega = f_1(x)dx^1 + f_2(x)dx^2 + ... + f_n(x)dx^n$$

i.e; $w = xdx + ydy$

"An object that eats a vector and spits out a number." (acts on a tangent vector)

2-Forms and higher:

A 2-form in $\mathbb{R}^{\nvDash}$ might look like:

$$d\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$$

Here,

- $\wedge$ is the **wedge product**, a way to "stack" differential elements.
- It is **anti-symmetric**: $dx \wedge dy = -dy \wedge dx$

These describe oriented areas, volumes, etc.

A.2 Wedge Product $\wedge$

If α is a 1-form and β is another 1-form, then:

$$\alpha \wedge \beta = -\beta \wedge \alpha \tag{A.1}$$

This captures orientation and area in higher dimensional space.

A.2.1 Workings of the Wedge Product

Taking two 1-forms on $\mathbb{R}^3$:

$$\alpha = f(x, y, z)dx$$

$$\beta = g(x, y, z)dy$$

Their wedge product is:

$$\alpha \wedge \beta = fg \ dx \wedge dy$$

This is a 2-form. Here, dx and dy are 1-forms. The wedge bilinear function of two tangent vectors.

$$\begin{aligned}(dx \wedge dy)(v, w) &= dx(v) \cdot dy(w) - dx(w) \cdot dy(v) \\ &= det \begin{bmatrix} dx(v) & dx(w) \\ dy(v) & dy(w) \end{bmatrix}\end{aligned}$$

Just the determinant of a 2x2 matrix.

Therefore, $dx \wedge dy$ measure the oriented area of the parallelogram spanned by v and w, projected onto the xy-plane.

Now, if we swap:

$$\beta \wedge \alpha = gf \ dy \wedge dx = -fg \ dx \wedge dy$$

We can wedge together up to n different dx^i's in $\mathbb{R}^n$; any repetations like $dx \wedge dx$ is zero, because:

$$dx \wedge dx = -dx \wedge dx \Rightarrow dx \wedge dx = 0 \tag{A.2}$$

A.3 Exterior Derivative (d)

This is a generalization of the gradient, curl, and divergence into a single operator.

If f is a 0-form (function), then:

$$df = \frac{df}{dx^i} dx^i \tag{A.3}$$

If ω is a 1-form, then $d\omega$ is a 2-form, like computing curl:

$$\text{Curl} \cdot \bar{F} \longleftrightarrow d\omega$$

Key Identity: $\mathbf{d^2}\omega = \mathbf{0}$ just like **curl·(grad(f)) = 0**

A.3.1 Workings of the Exterior Derivative

It can be said to be a differentiation that climbs up dimensions. This generalizes gradient, curl and divergence.

Let's say we have a 1-form in $\mathbb{R}^3$:

$$\omega = f(x,y,z)dx + g(x,y,z)dy + h(x,y,x)dz$$

This corresponds to a vector field $\bar{F} = (f, g, h)$

Applying the exterior derivative:

$$dw = d(fdx + gdy + hdz) = df \wedge dx + dg \wedge dy + dh \wedge dz \tag{A.4}$$

Now, for $\mathbf{df} \wedge \mathbf{dx}$:

$$\begin{aligned} df \wedge dx &= (\partial_x f \ dx + \partial_y f \ dy_{+} \partial_z f \ dz) \wedge dx \\ \Rightarrow df \wedge dx &= \partial_x f \ dx \wedge dx + \partial_y f \ dy \wedge dx + \partial_z f \ dz \wedge dx \\ \Rightarrow df \wedge dx &= \partial_y f \ dy \wedge dx + \partial_z f \ dz \wedge dx \end{aligned} \tag{A.5}$$

Similarly for $\mathbf{dg} \wedge \mathbf{dy}$:

$$dg \wedge dy = \partial_x g \ dy \wedge dx + \partial_z \ dy \wedge dz \tag{A.6}$$

and, $\mathbf{dh} \wedge \mathbf{dz}$:

$$dh \wedge dz = \partial_x h \ dx \wedge dz + \partial_y h \ dy \wedge dz \tag{A.7}$$

Plugging A.5, A.6 and A.7 into A.4:

$$\begin{aligned} dw &= \partial_y f \ dy \wedge dx + \partial_z f \ dz \wedge dx + \partial_x g \ dy \wedge dx + \partial_z \ dy \wedge dz + \partial_x h \ dx \wedge dz + \partial_y h \ dy \wedge dz \\ \Rightarrow dw &= (\partial_x g - \partial_y f) \ dx \wedge dy + (\partial_x - \partial_z f) \ dx \wedge dz + (\partial_y - \partial_z g) \ dy \wedge dz \end{aligned} \tag{A.8}$$

This corresponds to **curl** $\cdot\bar{\mathbf{F}}$.

Now, lets say we have a 2-form in $\mathbb{R}^{\nvDash}$:

$$dw = A(x,y,z) \ dy \wedge dz + B(x,y,z) \ dz \wedge dx + C(x,y,z) \ dx \wedge dy$$

$$d(dw) = dA(x,y,z) \ dy \wedge dz + dB(x,y,z) \ dz \wedge dx + dC(x,y,z) \ dx \wedge dy \tag{A.9}$$

$\underline{\mathbf{dA \wedge dy \wedge dz}:}$

$$(dA \wedge dy \wedge dz = \partial_x A dx + \partial_y A dy + \partial_z A dz) \wedge dy \wedge dz$$

$$\Rightarrow dA \wedge dy \wedge dz = \partial_x A\ dx \wedge dy \wedge dz + \partial_y\ dy \wedge dy \wedge dz + \partial_z\ dz \wedge dy \wedge dz$$

$$\Rightarrow dA \wedge dy \wedge dz = \partial_x A\ dx \wedge dy \wedge dz \tag{A.10}$$

Similarly we get,

$\underline{\mathbf{dB \wedge dz \wedge dx}:}$

$$dB \wedge dz \wedge dx = \partial_y B\ dx \wedge dy \wedge dz \tag{A.11}$$

$\underline{\mathbf{dC \wedge dx \wedge dy}:}$

$$dC \wedge dx \wedge dy = \partial_z C\ dx \wedge dy \wedge dz \tag{A.12}$$

Plugging (3.8), (3.9) and (3.10) in (3.7):

$$d^2 w = (\partial_x A + \partial_y B + \partial_z C)\ dz \wedge dy \wedge dz \tag{A.13}$$

Which corresponds to $\nabla \cdot \bar{F} = \partial_x A + \partial_y B + \partial_z C$

A.4 Hodge Product ($\star$)

In $\mathbb{R}^n$ with a metric and an orientation, the Hodge star is a map:

$$\star : \Lambda^k \to \Lambda^{n-k}$$

that takes a k-form and turns it into an $(n-k)$-form.

It is defined so that,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \text{vol} \tag{A.14}$$

Where, $\langle \alpha, \beta \rangle$ is the inner product of forms and vol is the oriented volume from $dx \wedge dy \wedge dz$ in $\mathbb{R}^3$.

A.4.1 How it acts in $\mathbb{R}^3$

In 3D, using the standard basis and orientation $dx \wedge dy \wedge dz$:

0-form to 3-form:

$$\star 1 = dx \wedge dy \wedge dz \tag{A.15}$$

1-form to 2-form:

$$\star dx = dy \wedge dz, \quad \star dy = dz \wedge dx, \quad \star dz = dx \wedge dy \tag{A.16}$$

2-form to 1-form:

$$\star(dy \wedge dz) = dx, \quad \star(dz \wedge dx) = dy, \quad \star(dx \wedge dy) = dz \tag{A.17}$$

3-form to 0-form:

$$\star(dx \wedge dy \wedge dz) = 1 \tag{A.18}$$

A.17 is just the "reverse" of A.16, because $\star$ is invertible and in Euclidean signature $\star\star\,\alpha = \alpha$ for 1-form and 2-form.

Form the example of 2-forms we saw earlier in section A.3.1:

$$\begin{aligned} dw &= A(x,y,z)\ dy \wedge dz + B(x,y,z)\ dz \wedge dx + C(x,y,z)\ dx \wedge dy \\ \Rightarrow dw &= A \star (dy \wedge dz) + B \star (dz \wedge dx) + C \star (dx \wedge dy) \\ \Rightarrow \star dw &= Adx + Bdy + Cdz \end{aligned} \tag{A.19}$$

In $\mathbb{R}^3$, we can interpret:

- A 1-form $Adx + Bdy + Cdz$ as the covector version of a vector field (A,B,C).
- A 2-form $Ady \wedge dx + Bdz \wedge dx + Cdx \wedge dy$ as the flux form associated with the same vector field.

so, the Hodge star acts like:

$$\text{(flux 2-form)} \overset{\star}{\longleftrightarrow} \text{(vector field 1-form)}$$

A.4.2 Why It Matters

- If we start with a flux 2-from, dw (encoding a magnetic field). d^2w is a 3-from representing divergence from language.
- If we want that divergence as a scalar, we apply $\star$ to dw:

$$\star dw = \nabla \cdot (A, B, C)$$

- If we start with a 1-form (say, an electric field), applying d then $\star$ gives us curl-type objects.

A.4.3 Workings of the Hodge Product

Taking a vector field $\bar{F} = (A, B, C)$ in $\mathbb{R}^3$ and representing it as a 2-form:

$$dw = Ady \wedge dx + Bdz \wedge dx + Cdx \wedge dy$$

This 2-form is the **Hodge dual**[6] of the 1-form:

$$\alpha = Adx + Bdy + Cdz$$

that is:

$$dw = \star\alpha \tag{A.20}$$

Now from taking the exterior derivative of dw, (3.11):

$$d^2w = (\partial_x A + \partial_y B + \partial_z C)\ dz \wedge dy \wedge dz$$

and applying the Hodge product:

$$\star dw = \star((\partial_x A + \partial_y B + \partial_z C)\ dz \wedge dy \wedge dz)$$
$$\Rightarrow \star dw = (\partial_x A + \partial_y B + \partial_z C) \star (\ dz \wedge dy \wedge dz)$$
$$\Rightarrow \star dw = (\partial_x A + \partial_y B + \partial_z C)(1)$$

$$\star dw = (\partial_x A + \partial_y B + \partial_z C) \tag{A.21}$$

(4.8) is exactly the divergence of the original vector, $A_x + B_y + C_z = \nabla \cdot (A, B, C)$

A.5 Maxwell's Equations

We want to express all four Maxwell's equations in vacuum using differential forms.

In standard vector calculus, Maxwell's equations in vacuum are:

1. **Gauss's Law**(Electric field):
$$\nabla \cdot \bar{E} = \frac{\rho}{\epsilon_0}$$
2. **Gauss's Law for Magnetism**:
$$\nabla \cdot \bar{B} = 0$$
3. **Faraday's Law**:
$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$
4. **Ampère-Maxwell Law**:
$$\nabla \times \bar{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$$

Now. we work on 4D Minkowski spacetime $\mathbb{R}^{1,3}$ and $\mu = 0, 1, 2, 3$.

[6] An isomorphism between k-forms and $(n-k)$-forms on an n-dimensional oriented manifold with a metric. This converts "volume in a k-dimensional subspace" into the complementary $(n-k)$-dimensional volume.

A.5.1 Defining the Field Strength 2-Form, F

In the component form we have:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{A.22}$$

where, A is the **4-potential**
the 4-potential in 1-from:

$$A = \phi dt - A_x dx - A_y dy - A_z dz \tag{A.23}$$

and the field strength tensor 2-form is:

$$F = dA \tag{A.24}$$

(5.20 and (5.3) combine $\bar{E}$ and $\bar{B}$ into one object:

$$F = E_x \ dx \wedge dt + E_y \ dy \wedge dt + E_z \ dz \wedge dt + B_x \ dy \wedge dz + B_y \ dz \wedge dx + B_z \ dx \wedge dy \tag{A.25}$$

Here, the electric part is $E \wedge dt$ and the magnetic part is a spatial 2-form.

Current as a 1-form:

$$J = \rho dt - J_x dx - J_y dy - J_z dz \tag{A.26}$$

and as a 3-form ($\star J$):

$$\star J = \rho \ dx \wedge dy \wedge dz - J_x \ dy \wedge dz \wedge dt - J_y \ dz \wedge dx \wedge dt - J_z \ dx \wedge dy \wedge dt \tag{A.27}$$

A.5.2 The Homogeneous Equations, dF = 0

For $F = dA$, we get $dF = d^2 A = 0$
Computing dF:
Considering the spatial 3-form;

$$dF = d^2 A = d(B_x \ dy \wedge dz + B_y \ dz \wedge dx + B_z . dx \wedge dy)$$

$$\Rightarrow dF = (\partial_x B_x + \partial_y B_y + \partial_z B_z) \ dx \wedge dy \wedge dz = 0 \tag{A.28}$$

Which corresponds to the form $\nabla \cdot \bar{B} = 0$ (Gauss's law for magnetism).

Now considering the mixed 2-form: time derivative of B and the spatial derivative of E,

$$dF = d(E_x \ dx \wedge dt + E_y \ dy \wedge dt + E_z \ dz \wedge dt) + d(B_x \ dy \wedge dz + B_y \ dz \wedge dx + B_z \ dx \wedge dy) = 0$$

$$\Rightarrow dF = (\partial_x E_y - \partial_y E_x + \partial_t B_z)\ dt\wedge dx\wedge dy + (\partial_x E_z - \partial_z E_x + \partial_t B_y)\ dt\wedge dx\wedge dz$$
$$+ (\partial_y E_z - \partial_z E_y + \partial_t B_x)\ dt\wedge dy\wedge dz + (\partial_x B_x + \partial_y B_y + \partial_z B_z)\ dx\wedge dy\wedge dz = 0$$

$$\Rightarrow dF = ((\nabla\times E)_z + \partial_t B_z)\ dt\wedge dx\wedge dy + ((\nabla\times E)_y + \partial_t B_y)\ dt\wedge dx\wedge dz$$
$$+ ((\nabla\times E)_x + \partial_t B_x)\ dt\wedge dy\wedge dz + (\nabla\cdot B)\ dx\wedge dy\wedge dz = 0$$

$$\Rightarrow dF = (\nabla\times E)_z\ dt\wedge dx\wedge dy + (\nabla\times E)_y\ dt\wedge dx\wedge dz + (\nabla\times E)_x\ dt\wedge dy\wedge dz$$
$$+ (\partial_t B_x)dt\wedge dy\wedge dz + (\partial_t B_y)dt\wedge dx\wedge dz + (\partial_t B_z)dt\wedge dx\wedge dy = 0 \qquad \text{(A.29)}$$

(5.8) corresponds to $\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$

A.5.3 The Inhomogeneous Equations, $\star\mathbf{d}\star\mathbf{F} = \mathbf{j}$

We had,

$$F = E_x\ dx\wedge dt + E_y\ dy\wedge dt + E_z\ dz\wedge dt + B_x\ dy\wedge dz + B_y\ dz\wedge dx + B_z\ dx\wedge dy$$

$$\star F = E_x\ dy\wedge\ dz + E_y\ dz\wedge dx + E_z\ dx\wedge dy + B_x\ dt\wedge dx + B_y\ dy\wedge dt + B_z\ dt\wedge dz$$

$$\Rightarrow d\star F = \partial_t E_x\ dt\wedge\ dy\wedge dz + \partial_x E_x dx\wedge dy\wedge dz + \partial_t E_y\ dt\wedge dx\wedge dz + \partial_y E_y dy\wedge dx\wedge dz$$
$$+ \partial_t E_z\ dt\wedge dx\wedge dy + \partial_z E_z dz\wedge dx\wedge dy$$

$$\Rightarrow d\star F = (\partial_z E_z + \partial_x B_y - \partial_z B_x)dt\wedge dx\wedge dy + (-\partial_t E_y + \partial_z B_x - \partial_x B_z)dt\wedge dx\wedge dz$$
$$+ (\partial_t E_x + \partial_z B_y - \partial_y B_z)dt\wedge dy\wedge dz + (\partial_x E_x + \partial_y E_y + \partial_z E_z)dx\wedge dy\wedge dz \qquad \text{(A.30)}$$

And now $\star d\star F$:

$$\star d\star F = (\partial_z E_z + \partial_x B_y - \partial_z B_x)dz + (\partial_t E_y - \partial_z B_x + \partial_x B_z)dy$$
$$+(\partial_t E_x + \partial_z B_y - \partial_y B_z)dx + (\partial_x E_x + \partial_y E_y + \partial_z E_z)dt \qquad \text{(A.31)}$$

Now,

$$(\partial_z E_z + \partial_x B_y - \partial_z B_x) = -J_z$$
$$(\partial_t E_y - \partial_z B_x + \partial_x B_z) = -J_y$$
$$(\partial_t E_x + \partial_z B_y - \partial_y B_z) = -J_x$$
$$(\partial_x E_x + \partial_y E_y + \partial_z E_z) = \rho$$

So, (5.10) becomes:

$$\star d \star F = -J_z dz - J_y dy - J_x dx + \rho dt$$

$$\Rightarrow \star d \star F = \rho dt - J_x dx - J_y dy - J_z dz = J \tag{A.32}$$

We get $\nabla \cdot \bar{E} = \frac{\rho}{\epsilon_0}$ and $\nabla \times \bar{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}$ from this.

B Integral of a Form

We can only integrate *top-degree* forms (degree = dimension of manifold) over an oriented manifold.

On a 4-dimensional manifold M, only 4-forms can be integrated over all of M. A 2-form can be integrated only over a 2-dimensonal surface inside M (e.g. $\int_S F$ in electromagnetism).

Integration sums values over each point. To sum meaningfully, we must assign a values to an infinitesimal volume element in that space, i.e. a top degree form.

An example would be: F-electromagnetic 2-form in 4D spacetime → integrate over 2D surface (worldsheet of a loop) to get flux.

B.1 Definition of Integrating a k-form

Let M be an n-dimensional oriented manifold. A k-form $\omega \in \Omega^k(M)$ is a smooth assignment of an alternating multilinear map.

$$\omega_p : (T_p M)^k \to \mathbb{R}$$

at each point p.

To integrate ω over a k-dimensional oriented submanifold $S \subset M$:

1. **We choose a coordinate chart $(\mathbf{U}, \phi)$ with local coordinates $(\mathbf{u^1}, ..., \mathbf{u^k})$ on S.** Meaning, if S is a k-dimensional oriented[7] submanifold of M, then locally it can be described by k-coordinates $(u^1, ..., u^k)$. The submanifold inherit these coordinates from M via the inclusion map $i : S \hookrightarrow M$.
   If $M = \mathbb{R}^3$ and S is the unit sphere S^2, a local coordinate chart would be spherical coordinate (θ, ϕ).

[7] *Oriented* here means that we have picked an order of the coordinate which will fix the sign of the integral.

2. **Pull ω back to $U \subset \mathbb{R}^k$.**

$$i * \omega = f(u) du^1 \wedge ... \wedge du^k \tag{B.1}$$

The k-form ω lives on M, nut we cannot integrate it over S until we restrict it to S. Mathematically, this is the **pullback**[8]:

$$i * \omega : S \to \Omega^k(S) \tag{B.2}$$

This essentially "feeds" ω only tangent vectors of S, ignoring any directions normal to S.
In coordinates $(u^1, ..., u^k)$ on S, the pullback has the general form:

$$i * \omega = f(u^1, ..., u^k) du^1 \wedge .... \wedge du^k \tag{B.3}$$

Here, f is the scalar density we will actually integrate.
If $w = x dy \wedge dz$ on $\mathbb{R}^3$, and S is the y-z plane at $x = 2$, then, $i * \omega = (2) dy \wedge dz$.

3. **Integrate the scalar function f over U using the usual k-dimensional Lebesgue integral[9]:**

$$\int_S \omega = \int_U f(u) du^1 ... du^k \tag{B.4}$$

Once we have $i * \omega = f(u) du^1 ... du^k$, we are in familiar territory – this is just the volume form of S multiplied by a function f.
Integrating ω over S means integrating f against the standard k-dimensional Lebesgue measure:

$$\int_S \omega = \int_{U \subset \mathbb{R}^k} f(u^1, ..., u^k) du^1 ... du^k \tag{B.5}$$

where U is the coordinate patch.
If S is curved, f will already include the Jacobian factor from the pullback[10].

Why this works?

This is just the change-of-variables theorem in disguise:

The form ω is intrinsically coordinate free. Pulling it back to coordinate (u^i) converts it to familiar *function* $\times$ *Volume element* form. Then integration reduces to the usual multivariable calculus.

[8] The inclusion map just "places" our smaller surface inside the bigger space and the pullbackkeeps only the parts of the form that lives along that surface so we can integrate it there.

[9] A k-dimensional Lebesgue integral is just a rigorous mathematical way of saying "integral over a k-dimensional region", where k could be 1 (line), 2 (surface), 3 (volume), etc. This is more general than Riemann integral – it can handle irregular shapes and functions with certain types of discontinuities.

[10] In smooth differential geometry, Lebesgue integral reduces to the usual multivariable integral with a Jacobian factor from a coordinate change.

Example

Let's integrate the 2-form:

$$w = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \tag{B.6}$$

Over the upper hemisphere of the unit sphere:

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0$$

1. **Parametrize the surface**
   Using spherical coordinates (θ, ϕ) with
   $$\begin{cases} \text{x=}\ \ \sin\theta\cos\phi \\ \text{y=}\sin\theta\sin\phi \\ \text{z=}\cos\theta \end{cases}$$
   where, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq 2\pi$
   This gives the inclusion map, $i : S \hookrightarrow \mathbb{R}^3$:
   $$i(\theta, \phi) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \tag{B.7}$$

2. **Compute the differentials**
   $$dx = \cos\theta\cos\phi\ d\theta - \sin\theta\sin\phi\ d\phi \tag{B.8}$$
   $$dy = \cos\theta\sin\phi\ d\theta + \sin\theta\cos\phi\ d\phi \tag{B.9}$$
   $$dz = -\sin\theta\ d\theta \tag{B.10}$$

3. **Pull back ω**

   Substituting into (1.6):

   $$x\ dy \wedge dz = \sin^3\theta\cos^2\phi\ (d\theta \wedge d\phi) \tag{B.11}$$
   $$y\ dz \wedge dx = \sin^3\theta\sin^2\phi\ (d\theta \wedge d\phi) \tag{B.12}$$
   $$z\ dx \wedge dy = \cos^2\theta\sin\theta\ (d\theta \wedge d\phi) \tag{B.13}$$

   Putting (1.11), (1.12), (1.13) into (1.6):
   $$\omega = \sin\theta\ d\theta \wedge d\phi \tag{B.14}$$

4. **Integrate over parameter domain**
   $$\int_S \omega = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\theta\ d\theta\ d\phi = 2\pi \tag{B.15}$$

B.2 Top-degree forms over the whole manifold

If dim$M = n$, then, to integrate over all of M, the form must be of degree n (a top-form[11]), so that its pullback to local coordinates has the form:

$$f(x)dx^1 \wedge ... \wedge dx^n \tag{B.16}$$

which corresponds to an infinitesimal volume element.

If w has degree $k > n$, it can still be integrated, but only over k-dimensional oriented submanifold of M (like a curve for $k = 1$, a surface for $k = 2$, etc.).

Mthematically, the pairing

$$\int_{S^k} \omega \tag{B.17}$$

only makes sense if $deg\ \omega = dim\ S^k$.

In short:

Only **top degree** forms can be integrated over the whole manifold, because the degree of the form must match the dimension of the space we are integrating over.

B.2.1 Why only Top-degree Forms

Let M be n-dimensional, oriented, with local coordinates $(x^1, ..., x^n)$.

A k-form ω (with $k < n$) looks locally like:

$$\omega = \sum_{|I|=k} f_I(x) dx^{i_1} \wedge ... \wedge dx^{i_k} \tag{B.18}$$

To integrate ovel all of M, we need a top form (an n-form) of the shape:

$$g(x)dx^1 \wedge ... \wedge dx^n \tag{B.19}$$

because that is the coordinate-free version of a volume element.

A k-form with $k < n$ simply does not contain enough differentials to produce such a volume element.

So, to summarize, we cannot integrate a k-form over an n-manifold when $k < n$ without adding extra, non-canonical structure. The only intrinsic, coordinate-invariant integral over the whole manifold is for top-degree forms.

B.3 Integrating a 4-form on $\mathbf{M} = \mathbb{R}^4$

Let,

$$\Omega = \rho(t, x, y, z)\ dt \wedge dx \wedge dy \wedge dz \tag{B.20}$$

[11] A differential form of the highest possible degree on a manifold.

using global coordinates (t,x,y,z) and the standard orientation $dt \wedge dx \wedge dy \wedge dz$. Here, the chart is the identity[12], so the pullback is trivial:

$$(id)^*\Omega = \rho(t,x,y,z)\ dt \wedge dx \wedge dy \wedge dz \tag{B.21}$$

Now, integrating over a bounded region ($\mathcal{U} \subset \mathbb{R}^4$):

$$\int_{\mathcal{U}} \Omega = \int_{\mathcal{U}} \rho(t,x,y,z)\ dt\ dx\ dy\ dz \tag{B.22}$$

Over all of M, we need ρ to decay sufficiently fast so the integral diverges. Moreover, only 4-forms look like a volume element $(function) \times dt \wedge dx \wedge dy \wedge dz$. Lower degree forms do not provide a 4D volume density.

B.4 Integrating a 2-form over a 2D surface $\mathbf{S^2} \subset \mathbf{M}$

Let the electromagnetic 2-form be:

$$F = E_i dx^i \wedge dt + \frac{1}{2}\epsilon_{ijk} B_k dx^i \wedge dx^j \tag{B.23}$$

We can integrate F over any oriented 2D submanifold S^2 in spacetime. As shown in the example earlier:

1. parametrize S by coordinates $u = (u^1, u^2)$: an inclusion map$i : S \hookrightarrow M$,

$$x^\mu = x^\mu(u^1, u^2) \ \ (\mu = 0,1,2,3 \text{ for } t,x,y,z)$$

2. Differentiate: $dx^\mu = \partial_{\mu^a} x^\mu du^a$ (sum over a = 1,2).

3. Pull back: Substitute $x^\mu(u), dx^\mu(u)$ into F to get $i^*F = f(u)du^1 \wedge du^2$.

4. Integrate: $\int_{S^2} F = \int_U f(u)du^1 du^2$. Which in physics gives electric or magnetic flux through S^2.

Physical Analogy

In electromagnetism:

- F(2-form) integrated over a spatial 2-surface = magnetic flux through that surface.
- $\star j$(3-form) integrated over a 3-dimensional volume = total charge in that volume.
- Only 4-forms can be integrated over all of spacetime, to get one from F, we need to wedge it with another 2-form or take $F \wedge \star F$ making it a 4-form.

[12] If instead we had a submanifold (like a surface or curve), the inclusion map would not be the identity – it would substitute the parametric equations into the form and kill irrelevant terms. That's when the pullback actually changes form.

C Proofs

Proof

C.1 Proof of $\mathbf{d^2\omega_p = 0}$

In a chart $(x^1, ..., x^n)$, any p-form looks like:

$$\omega = \frac{1}{p!}\omega_{i_1....i_p}(x)dx^{i_1} \wedge .... \wedge dx^{i_p} \tag{C.1}$$

<u>Applying the first exterior derivative:</u>

By definition:

$$d\omega = \frac{1}{p!}\partial_j \omega_{i_1}....\omega_{i_p}(x)dx^j \wedge dx^{i_1} \wedge ... \wedge dx^{i_p} \tag{C.2}$$

The new form has degree $p+1$. The wedge $dx^j \wedge dx^{i_1} \wedge .....$ is antisymmetric automatically. If we swap two indices, then the whole wedge changes sign:

$$dx^{i_1} \wedge dx^j = -dx^j \wedge dx^{i_1} \tag{C.3}$$

If the two indices are equal $(j = i_1)$, then the product vaniahes,

$$dx^j \wedge dx^j = 0 \tag{C.4}$$

<u>Applying the second exterior derivative:</u>

$$d^2\omega = \frac{1}{p!}\partial_l \partial_j \omega_{i_1}...\omega_{i_p} dx^l \wedge dx^j \wedge dx^{i_1} \wedge ... \wedge dx^{i_p} \tag{C.5}$$

Now. partial derivatives are symmetric and the wedge products are antisymmetric. If we swap indices (j, l), the coefficient stays the same (symmetry), but the wedge basis picks up a minus:

$$\partial_j \partial_l \omega_{i_1....i_p} dx^j \wedge dx^l = \partial_l \partial_j \omega_{i_1....i_p}(dx^l \wedge dx^j) \tag{C.6}$$

Every pair (l, j) appears twice in the sum, once as (l, j), once as (j, l). Each pair cancels out. Therefore,

$$d^2\omega = 0 \tag{C.7}$$

Proof

C.2 Proving $\star\star\alpha = (-1)^{p(n-p)}\alpha$

Let (M,g) be an oriented Riemann n-manifold. Fixing a point and choosing an oriented orthonormal coframe, $\{e^1, ..., e^n\}$ so the volume form is:

$$\text{vol} = e^1 \wedge .... \wedge e^n \tag{C.8}$$

Let, $e^{i_1} \wedge .... \wedge e^{i_p} := e^I$ and the complement, e^J. So,

$$e^I \wedge e^J = \text{vol} \tag{C.9}$$

By definition of $\star$, takes a p-form and gives a (n-p)-form,

$$e^I \wedge \star e^J = \text{vol} \tag{C.10}$$

Here, $\star e^I$ must be the wedge of the missing basis elements upto a sign. So,

$$\star e^I = \pm e^J \tag{C.11}$$

Now, applying $\star$ again,

$$\star\star e^I = \pm\star e^J = \pm\pm e^I \tag{C.12}$$

So,

$$\star\star e^I = \pm(\pm e^I) \tag{C.13}$$

$e^J \to e^I$ has index (n-p) and since it is a p-form, $\star\star e^I$ has to cross (n-p), p times: $(-1)^{p(n-p)}$ would determine the sign depending on the values of p and n. So, we have:

$$\star\star e^I = (-1)^{p(n-p)} e^I \tag{C.14}$$

Proof

C.3 $\star d \star \alpha$ gives divergence

The 1-form α is written as:

$$\alpha = g_{ij} X^i dx^j \tag{C.15}$$

The Hodge star on 1-form is:

$$\star(dx^j) = \frac{\sqrt{|g|}}{(n-1)!} g^{jk} \epsilon_{ki_2....i_n} dx^{i_2} \wedge ... \wedge dx^{i_n} \tag{C.16}$$

So,

$$\star \alpha = g_{ij} X^i \star (dx^j)$$

$$\Rightarrow \star\alpha = g_{ij} X^i \frac{\sqrt{|g|}}{(n-1)!} \epsilon_{ki_2....i_n} dx^{i_2} \wedge ... \wedge dx^{i_n}$$

$$\Rightarrow \star\alpha = X^k \frac{\sqrt{|g|}}{(n-1)!} \epsilon_{ki_2....i_n} dx^{i_2} \wedge ... \wedge dx^{i_n}$$

We then have:

$$\star\alpha = X^k \frac{\sqrt{|g|}}{(n-1)!} \epsilon_{ki_2....i_n} dx^{i_2} \wedge ... \wedge dx^{i_n} \tag{C.17}$$

Taking the exterior derivative we get,

$$d(\star\alpha) = \frac{1}{(n-1)!} \partial_i(\sqrt{|g|}\ X^k) \epsilon_{ki_2....i_n} dx^1 \wedge dx^{i_2} \wedge .... \wedge dx^{i_n} \tag{C.18}$$

By definition:

$$dx^i \wedge dx^{i_2} \wedge ... \wedge dx^{i_n} = \epsilon_{ii_2....i_n} d^n x \tag{C.19}$$

By inputting C.19 into C.18 we get,

$$d(\star\alpha) = \frac{1}{(n-1)!} \partial_i(\sqrt{|g|}\ X^k) \epsilon_{ki_2....i_n} \epsilon_{ii_2....i_n} d^n x \tag{C.20}$$

Now using the identity $\epsilon_{kI}\epsilon_{iI} = (n-1)!\delta_i^k$ where, $I = (i_2, ..., i_n)$:

$$d(\star\alpha) = \frac{1}{(n-1)!} \partial_i(\sqrt{|g|}\ X^k)[(n-1)!\ \delta_i^k]\ d^n x = \partial_i(\sqrt{|g|}\ X^i)\ d^n x \tag{C.21}$$

Now,

$$\text{Vol} = \sqrt{|g|}\ dx^1 \wedge ... \wedge dx^n = \sqrt{|g|}\ d^n x$$

and plugging this into with a little bit of algebra:

$$d(\star\alpha) = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\ X^i)\ \sqrt{|g|}\ d^n x$$

$$\Rightarrow d(\star\alpha) = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\ X^i)\ \text{Vol} \tag{C.22}$$

Applying hodge star on C.22:

$$\star d \star \alpha = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\ X^i) \star \text{Vol} \tag{C.23}$$

Now, we know:

$$\star 1 = \sqrt{|g|}\ dx^1 \wedge ... \wedge dx^n = \text{Vol}$$

and previously proved $\star\star\alpha$ gives us α upto a sign:

$$\star d \star \alpha = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\ X^i) \star \text{Vol} = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\ X^i)(1)$$

Finally,

$$\star d \star \alpha = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\ X^i) \tag{C.24}$$

Which is exactly the **divergence density** of the vector field X in curved spacetime. In flat spacetime $\sqrt{|g|} = 1$. So, we have:

$$\star d \star \alpha = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\ X^i) = \partial_i X^i \tag{C.25}$$

and reduces down to $\nabla \cdot X$.

Proof

C.4 Proving $\langle \mathbf{d}\eta, \omega \rangle = \langle \eta, \mathbf{d}^{\dagger}\omega \rangle$

We know,

$$\langle d\eta, \omega \rangle = \int_M d\eta \wedge \star\omega \tag{C.26}$$

Using the graded Leibniz rule (for $deg\ \eta = p - 1$):

$$d(\eta \wedge \star\omega) = d\eta \wedge \star\omega + (-1)^{p-1}\ \eta \wedge d(\star\omega) \tag{C.27}$$

So, $d\eta \wedge \star\omega$ from C.27 is:

$$d\eta \wedge \star\omega = d(\eta \wedge \star\omega) - (-1)^{p-1}\eta \wedge d(\star\omega) \tag{C.28}$$

So, plugging C.28 into C.26 gives us:

$$\langle d\eta, \omega \rangle = \int_M d\eta \wedge \star\omega = \int_M d(\eta \wedge \star\omega) - \int_M (-1)^{p-1}\eta \wedge d(\star\omega) \tag{C.29}$$

Applying Stoke's theorem $\int_M d(\eta \wedge \star\omega) = \int_{\partial M} i(\eta \wedge \star\omega)$:

$$= \int_{\partial M} i(\eta \wedge \star\omega) - (-1)^{p-1} \int \eta \wedge d \star \omega \tag{C.30}$$

Applying appropriate boundary conditions:

$$\langle d\eta, \omega \rangle = -(-1)^{p-1} \int \eta \wedge d(\star\omega) \tag{C.31}$$

Now, considering $-(-1)^{p-1}$:

For, $p - 1$ even, $(-1)^{p-1} = +1$ and $-(+1) = -1$.

For, $p - 1$ odd, $(-1)^{p-1} = -1$ and $-(-1) = +1$.

For, p even, $(-1)^{p-1} = +1$ and $-(+1) = -1$.

For, p odd, $(-1)^{p-1} = -1$ and $-(-1) = +1$.

So, $-(-1)^{p-1} = (-1)^p$
We then have,

$$\int_M d\eta \wedge \star\omega = (-1)^p \int_M \eta \wedge d(\star\omega) \tag{C.32}$$

Form the definition of co-derivatives:

$$d^\dagger = (-1)^{n(p+1)+1} \ \star d \star$$

So,

$$d^\dagger \omega = (-1)^{n(p+1)+1} \star d \star \omega \tag{C.33}$$

Performing on ω:

- $\star \to (n-p)$-form
- $d \to (n-p+1)$-form
- $\star \to (p-1)$-form

So, the sign factor would be

$$(-1)^{n(p+1)+1(p-1)} = (-1)^{p(n-p)}$$

Now, following from C.33:

$$\star d^\dagger \omega = (-1)^{n(p+1)+1} \star \star d \star \omega \tag{C.34}$$

Coming down from C.14:

$$\star \star d \star \omega = (-1)^{p(n-p)} d \star \omega$$

So, we get:

$$\star(d^\dagger \omega) = (-1)^{n(p+1)+1} (-1)^{p(n-p)} d \star \omega$$

$$\Rightarrow \star(d^\dagger \omega) = (-1)^{n(p+1)+1p(n-p)} d \star \omega$$

$$\Rightarrow d \star \omega = (-1)^{n(p+1)+1p(n-p)} \star (d^\dagger \omega) = \langle \eta, d^\dagger \omega \rangle \tag{C.35}$$

So, we end up with, coming down from C.29:

$$\langle d\eta, \omega \rangle = \langle \eta, d^\dagger \omega \rangle \tag{C.36}$$

Proof

C.5 Action of a p-form field strength and equations of motion

The action:

$$S = -\frac{1}{2g^2}\int_M F \wedge \star F = -\frac{1}{2g^2}\int_M \sqrt{|g|}\frac{1}{p!}F_{\mu_1\ldots\mu_p}F^{\mu_1\ldots\mu_p}\ d^n x \tag{C.37}$$

Gauge invariance: $A \rightarrow A + d\Lambda$, $(\Lambda \in \Omega^{p-2})$ leaves $F = dA$ and thus S remains unchanged:
This is the gauge symmetry of a $p-$form field. By definition the field strength is $F = dA$ and $dF = d^2 A = 0$, Applying the gauge transformation: $A' = A + d\Lambda$:

$$F' = dA' = d(A + d\Lambda) = dA + d^2\Lambda = dA = F \tag{C.38}$$

Now, varying, $\delta F = d\delta A$
so,

$$\delta S = -\frac{1}{2g^2}\int_M d(\delta A)\wedge \star F$$

$$= -\frac{1}{2g^2}\bigg(\int_M d(\delta A \wedge \star F) - (-1)^{p-1}\int_M \delta A \wedge d\star F\bigg)$$

$$= -\frac{1}{2g^2}\bigg(\int_{\partial M} i(\delta A \wedge \star F) - (-1)^{p-1}\int_M \delta A \wedge \star F\bigg)$$

With appropriate boundary conditions(similar to how we got (4.6)):

$$= -\frac{1}{2g^2}\bigg(-(-1)^{p-1}\int_M \delta A \wedge d\star F\bigg) \tag{C.39}$$

Now, $\delta S = 0$ and for (5.3) to be zero and the integral to vanish (δA is arbitrary inside M):

$$d\star F = 0 \tag{C.40}$$

Now, adding source term $S_s = \int_M A \wedge \star J$, and varying $\delta S_s = \int_M \delta A \wedge \star J$, we then have:

$$\delta = \frac{(-1)^{p-1}}{2g^2}\int_M \delta A \wedge d(\star F) + \int_M \delta A \wedge \star J \tag{C.41}$$

In this case, we require:

$$\frac{(-1)^{p-1}}{2g^2}d(\star F) + \star J = 0$$

$$\Rightarrow d(\star F) = (-1)^{p-1} 2g^2 \star J \tag{C.42}$$

Absorbing:

$$d(\star F) = \star J \tag{C.43}$$

C.40 and C.43 are the equations of motion.

In component form:

$$(d \star F)_{\mu_2 \ldots \mu_p} = \nabla^\nu F_{\nu \mu_2 \ldots \mu_p} \tag{C.44}$$

So, the equations of motion are:

$$\nabla_\nu F^{\nu \mu_2 \ldots \mu_p} = J^{\mu_2 \ldots \mu_p} \tag{C.45}$$

For, $p = 2$ (Maxwell in n=4):

$$\nabla_\nu F^{\nu \mu_2} = J^{\mu_2} \tag{C.46}$$

Which is the standard inhomogeneous Maxwell's equation.

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